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Homological Algebra Modulo a Regular Sequence with Special Attention to Codimension Two

Luchezar L. Avramov[1]

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail: avramov@math.purdue.edu

and

Ragnar-Olaf Buchweitz[2]

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada
E-mail: ragnar@math.utoronto.ca

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Let M be a finite module over a ring R obtained from a commutative ring Q by factoring out an ideal generated by a regular sequence. The homological properties M over R and over Q are intimately related. Their links are analyzed here from the point of view of differential graded homological algebra over a Koszul complex that resolves R over Q. One outcome of this approach is a transparent derivation of some central results of the theory. Another is a new insight into codimension two phenomena, yielding an explicit finitistic construction of the generally infinite minimal R-free resolution of M. It leads to theorems on the structure and classification of finite modules over codimension two local complete intersections that are exact counterparts of Eisenbud's results for modules over hypersurfaces.

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CONTENTS

INTRODUCTION

Regular sequences appeared simultaneously in two landmark papers published in 1956 at the outset of the "homological era" [24] of commutative ring theory.

Auslander and Buchsbaum introduced regular sequences in [1] under the present name and used them to prove that regular local rings have finite global dimension; details were given in [2]. Without naming them Rees [29] put regular sequences at the foundation of his theory of grade for ideals and its extension to modules in [30]. Since then, a number of authors have returned to the general "change-of-rings" question:

How is the module theory of a commutative ring Q related to that of its residue ring $R = Q/(\boldsymbol{f})$ when $\boldsymbol{f} = f_1, \dots, f_c$ is a Q-regular sequence?

To put our results in context we provide a short historical overview.

Rees [29] provided part of the answer: If an R-module M *lifts to* Q, in the sense that $M \cong L/(\boldsymbol{f})L$ for a Q-module L on which $\boldsymbol{f}$ is regular, then the homological properties of M over R are determined by those of L over Q.

Tate in 1957 constructed an R-free resolution $\mathbf{F}$ of $Q/(\boldsymbol{g})$ when $\boldsymbol{g}$ is a Q-regular sequence and $\boldsymbol{f} \subset (\boldsymbol{g})$. His paper [33] introduced to commutative algebra the technique of DG (= differential graded) algebra resolutions, a version of the "constructions" developed by H. Cartan for his celebrated computation of the homology of Eilenberg–MacLane spaces [15]. When R is a local complete intersection, meaning that Q can be taken to be a regular local ring, Tate's construction yields a minimal free resolution of the residue field k of R.

In 1969 Shamash [31] significantly extended Tate's result in the case of codimension 1, that is, when the regular sequence consists of a single element. Starting from a Q-free resolution $\mathbf{E}$ of an R-module M, he produced inductively a system σ of "higher homotopies" on $\mathbf{E}$, fitted them together to get a differential on a free graded module with $F_n = \bigoplus_{u \geqslant 0} R \otimes_Q E_{n-2u}$, and showed that the resulting complex $\mathbf{E}\{\boldsymbol{\sigma}\}$ is a free resolution of M. As a consequence, the Betti numbers $\beta_n^R(M) = \operatorname{rank}_k \operatorname{Tor}_n^R(M, k)$ of any finite module M over a local complete intersection R are bounded above by a polynomial in n. If $d - 1$ is the minimal degree of such a polynomial M is (nowadays) said to have complexity d, denoted $\operatorname{cx}_R M = d$.

Gulliksen brought cohomology into the picture in 1974. The new tool in [21] is an action of a graded polynomial algebra $\mathcal{S} = R[\chi_1, \ldots, \chi_c]$ on $\operatorname{Ext}_R^*(M, N)$ and on $\operatorname{Tor}_*^R(M, N)$, natural in both M and N. His operators χ_j of cohomological degree 2 are defined in terms of connecting homomorphisms. When rings and modules are noetherian and the projective dimension of M over Q is finite $\operatorname{Ext}_R^*(M, N)$ is a noetherian $\mathcal{S}$-module and $\operatorname{Tor}_*^R(M, N)$ is a (relatively) artinian one. This result opened the road from commutative to homological algebra to two-way traffic.

In 1980, Eisenbud [18] took that road to study the case when Q is a regular local ring. He proved that if $\operatorname{cx}_R M \leq 1$ then the minimal free resolution of M becomes periodic of period 2 after at most (depth $R + 1$) steps; in codimension 1 he expressed the periodic tail from a "matrix factorization" of the defining equation f. His key instruments were $\mathcal{S}$-module structures on (co)homology induced by chain maps of degree -2 that operate on any free resolution of M and commute up to homotopy. Extending Shamash's construction to arbitrary codimension, he produced a not necessarily minimal R-free resolution $\mathbf{E}\{\boldsymbol{\sigma}\}$ of M that is a DG module over $\mathcal{S}$.

More recently, results on modules over complete intersections have led to the introduction of new homological dimensions for modules over commutative rings. In [5, 9] most results on modules over complete intersections have been extended to modules of finite CI-dimension over arbitrary local rings. The Betti sequence of such modules has been shown to grow asymptotically as a polynomial in n. When $\operatorname{cx}_R M \leq 1$, the sequence stabilizes after at most depth $R + 1$ steps; when $\operatorname{cx}_R M \geq 2$, it eventually strictly increases.

Next we describe the material in this paper.

In Section 1 we recall a few basic facts from DG homological algebra over the Koszul complex $\mathbf{K}$ resolving R over Q; complete details may be found in [7, Sect. 1]. The upshot is that if $R \leftarrow Q \rightarrow k$ are commutative ring homomorphisms, then $\operatorname{Ext}_Q^*(M, k)$ is naturally a graded module over $\operatorname{Tor}_*^Q(R, k)$. This is the cohomological counterpart of the classical homological product of Cartan and Eilenberg [16].

Starting from a Q-free resolution $\mathbf{E}$ of M with action of the Koszul complex $\mathbf{K}(\boldsymbol{f}; Q)$, we produce in Section 2 a universal R-free resolution $\mathrm{G}(\mathbf{E})$ of M. It seems to be simpler to construct and to analyze than Tate's resolution, which it generalizes, or that of Shamash and Eisenbud, of which it is a special case. Its explicit structure of DG module over $\mathcal{S}$ is used in Section 3 to introduce the operators in cohomology and to compute Exts in several cases of interest.

To describe the results of Section 4 we assume for simplicity that Q is a regular local ring with algebraically closed residue field k and M is a maximal Cohen–Macaulay module over a complete intersection $R = Q/(f_1, f_2)$ of codimension 2. The resolution $\mathrm{G}(\mathbf{E})$ yields $\mathrm{Ext}_R^*(M, k)$ as the homology of a natural linear complex $\mathscr{C}^\bullet(M, k)$ of length two whose terms are free graded modules over the polynomial ring $\mathscr{R} = k[\chi_1, \chi_2]$. That complex is obtained by a standard procedure from the graded module $\mathsf{M} = \mathrm{Ext}_Q^*(M, k)$ over the exterior algebra $\mathsf{T} = \mathrm{Tor}_*^Q(R, k)$ on two variables of degree 1. The indecomposable direct summands of M are determined by Kronecker's theory of pencils of matrices and provide a decomposition of $\mathscr{C}^\bullet(M, k)$ into a direct sum of explicitly known linear complexes. All this allows description in detail of the $\mathscr{R}$-module $\mathrm{Ext}_R^{\geqslant n}(M, k)$ for $n > m = 2\max\{\beta_0^R(M), \beta_1^R(M)\}$.

Using the computation above, we trim in Section 5 the universal resolution of the mth syzygy of M in order to get a minimal R-free resolution $\mathbf{F}$ of M. The process shows that the truncation $\mathbf{F}_{>m}$ has a structure of DG module over the ring of operators $\mathscr{S} = R[\chi_1, \chi_2]$. This proves in codimension 2 a conjecture of Eisenbud [18] that remains completely open in higher codimensions.

In Section 6 we return to a general ring Q and its residue ring R modulo a regular sequence $\boldsymbol{f} = f_1, \ldots, f_c$. We set up a change-of-rings spectral sequence converging to $\mathrm{Ext}_R^*(M, N)$, when M and N are modules over R. Its second page is a linear complex $\mathscr{C}^\bullet(M, N)$ of graded $\mathscr{S}$-modules, determined by the graded module $\mathrm{Ext}_Q^*(M, N)$ over the exterior algebra $R\langle \xi_1, \ldots, \xi_c \rangle = \mathrm{Tor}_*^Q(R, R)$. The sequence yields a short transparent proof of Gulliksen's finiteness theorem.

In the final Section 7 we focus on a module M of finite CI-dimension over a local ring. From the spectral sequence we derive a new proof that the Betti sequence of M grows asymptotically like a polynomial. From the results in codimension 2 we obtain a realistic estimate of that place in the resolution of a module of complexity 2 beyond which the Betti numbers start to increase.

1. HOMOLOGY OPERATORS

In this section we prove the following two results.

1.1. THEOREM. *If $R \leftarrow Q \rightarrow k$ are homomorphisms of commutative rings, then $\mathsf{T} = \mathrm{Tor}_*^Q(R, k)$ is a graded associative and commutative algebra through pairings*

$$\mathrm{Tor}_i^Q(R, k) \otimes_Q \mathrm{Tor}_j^Q(R, k) \rightarrow \mathrm{Tor}_{i+j}^Q(R, k)$$

that extend the canonical product on $\mathrm{Tor}_0^Q(R, k) = R \otimes_Q k$.

For any R-module M and any k-module N there are pairings

$$\operatorname{Tor}_i^Q(R,k) \otimes_Q \operatorname{Tor}_j^Q(M,N) \to \operatorname{Tor}_{i+j}^Q(M,N),$$
$$\operatorname{Tor}_i^Q(R,k) \otimes_Q \operatorname{Ext}_Q^j(M,N) \to \operatorname{Ext}_Q^{j-i}(M,N)$$

that endow $\operatorname{Tor}_*^Q(M,N)$ *and* $\operatorname{Ext}_Q^*(M,N)$ *with a structure of graded module over* T, *extending the standard actions of* $R \otimes_Q k$ *on* $M \otimes_Q N$ *and on* $\operatorname{Hom}_Q(M,N)$.

These actions of T *are natural in M and N; exact sequences of R-modules or exact sequences of k-modules induce* T*-linear connecting morphisms.*

Each k-module N′ defines a natural morphism of graded $\operatorname{Tor}_*^Q(R,k)$*-modules*

$$\operatorname{Ext}_Q^*(M, \operatorname{Hom}_k(N,N')) \to \operatorname{Hom}_k(\operatorname{Tor}_*^Q(M,N), N')$$

that is an isomorphism whenever the k-module N′ is injective.

The algebra and module structures on Tor's are classical, given by the homology product $\pitchfork$ of Cartan and Eilenberg [16].

As a first application of the theorem we show that the action of Tor detects free summands in certain cases. The largest rank of a free direct summand of a module L over a ring A is known as the *free rank* of L, denoted $\operatorname{f-rank}_A L$.

Recall that a finite module $M \neq 0$ over a local (noetherian) ring Q is said to be *perfect* if its projective dimension $\operatorname{proj\,dim}_Q M$ equals the length of a maximal Q-regular sequence contained in its annihilator. An ideal $\mathfrak{a} \subset Q$ is *Gorenstein* of codimension c if $Q/\mathfrak{a}$ is a perfect Q-module whose minimal free resolution $\mathbf{P}$ has length c and satisfies $P_c \cong Q$.

1.2. Proposition. *Let* $\mathfrak{a}$ *be a Gorenstein ideal of codimension c in a local ring Q with residue field k; set* $R = Q/\mathfrak{a}$ *and* $\mathsf{T} = \operatorname{Tor}_*^Q(R,k)$. *If M is a finite R-module with* $\operatorname{proj\,dim}_Q M < \infty$ *and* $\operatorname{depth} M = \operatorname{depth} R$, *then*

$$\operatorname{f-rank}_R M = \operatorname{f-rank}_{\mathsf{T}} \operatorname{Ext}_Q^*(M,k) = \operatorname{f-rank}_{\mathsf{T}} \operatorname{Tor}_*^Q(M,k).$$

The notation of Theorem 1.1 remains in force throughout the section. We describe next some basic DG algebra techniques used in various proofs of the paper.

1.3

The components of a graded Q-module L are indexed either by homological degree, written as a subscript, or by cohomological degree, appearing as a superscript. These degrees are interchangeable by the rule $L^i = L_{-i}$ for each $i \in \mathbb{Z}$; for $u \in L_i$ or $u \in L^i$, we set $|u| = |i|$. If L and L' are graded (left) modules over a graded algebra A, then a homomorphism $\lambda\colon L \to L'$

of homological degree d is a family of additive maps $\lambda_i: L_i \to L'_{i+d}$ for $i \in \mathbb{Z}$, such that $\lambda(au) = (-1)^{di} a\lambda(u)$ for all $u \in L$, $a \in A_i$, and $i \in \mathbb{Z}$; note that λ has cohomological degree $-d$. Homomorphisms of graded A-modules, of arbitrary degree, are also referred to as A-linear maps; degree zero homomorphisms are called morphisms.

1.4

For each $s \in \mathbb{Z}$ we denote $L[s]$ the graded abelian group with $L[s]_i = L_{i-s}$, or, equivalently, $L[s]^i = L^{i+s}$ for all $i \in \mathbb{Z}$. Let $\Sigma^s: L \to L[s]$ be the map $L \to L[s]$ of homological degree s defined by $\Sigma^s(u) = u$ for all u, and turn $L[s]$ into a graded A-module by setting $a\Sigma^s(u) = (-1)^{si}\Sigma^s(au)$ for $a \in A_i$ and $u \in L$. In this way Σ^s becomes a degree s homomorphism of left graded A-modules.

For instance, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of Q-modules, then the connecting maps $\partial^n: \mathrm{Ext}^n_Q(M', N) \to \mathrm{Ext}^{n+1}_Q(M'', N)$ define a *connecting morphism* of graded Q-modules $\partial: \mathrm{Ext}^*_Q(M', N)[-1] \to \mathrm{Ext}^*_Q(M'', N)$. Similarly, an exact sequence $0 \to N' \to N \to N'' \to 0$ defines a connecting morphism $\partial: \mathrm{Ext}^*_Q(M, N'')[-1] \to \mathrm{Ext}^*_Q(M, N')$.

1.5

The graded module underlying a complex $\mathbf{U}$ is denoted $\mathbf{U}^\natural$. A *morphism* of complexes $\alpha: \mathbf{U} \to \mathbf{U}'$ is a morphism of graded Q-modules that satisfies $\alpha\partial = \partial\alpha$. A *quasi-isomorphism* is a morphism such that $\mathrm{H}_i(\alpha)$ is an isomorphism for all i.

A DG algebra $\mathbf{K}$ is a complex whose underlying graded module is a graded algebra and whose differential satisfies the Leibniz rule $\partial(ab) = \partial(a)b + (-1)^i a\partial(b)$ for $a \in \mathbf{K}_i$ and $b \in \mathbf{K}$. Similarly, a DG module $\mathbf{U}$ over $\mathbf{K}$ is a complex such that $\mathbf{U}^\natural$ is a graded $\mathbf{K}^\natural$-module and $\partial(au) = \partial(a)u + (-1)^i a\partial(u)$ for $a \in \mathbf{K}_i$ and $u \in \mathbf{U}$.

Any ring can and will be considered a DG algebra concentrated in degree zero.

1.6. EXAMPLE. Let $\boldsymbol{f} = \{f_1, \ldots, f_c\}$ be a set of elements of Q.

The *Koszul complex* $\mathbf{K}(\boldsymbol{f}; Q)$ is the DG algebra with $\mathbf{K}(\boldsymbol{f}; Q)^\natural$ the exterior algebra on a free Q-module with basis $\xi_1, \ldots, \xi_c$ of homological degree 1, and differential ∂ such that $\partial(\xi_j) = f_j$ for $j = 1, \ldots, c$. It can also be described as the DG algebra obtained from Q by adjoining *exterior variables* $\xi_1, \ldots, \xi_c$ to kill the cycles $f_1, \ldots, f_c$; cf. [7, Sect. 6.1, 33]. One then writes $\mathbf{K}(\boldsymbol{f}; Q) = Q\langle \xi_1, \ldots, \xi_s \mid \partial(\xi_j) = f_j\rangle$.

We say that a complex of Q-modules $\mathbf{E}$ has a *Koszul structure* if it is a DG module over $\mathbf{K}(\boldsymbol{f}; Q)$. By the Leibniz rule $\partial(\xi_j e) + \xi_j\partial(e) = f_j e$ for all $e \in \mathbf{E}$, so the map $\sigma^{(j)}(e) = \xi_j e$ of degree 1 is a Q-linear homotopy from

$f_j\,\mathrm{id}^{\mathbf{E}}$ to $0^{\mathbf{E}}$. Thus, a Koszul structure on $\mathbf{E}$ is nothing but a choice of a family of homotopies $\left(\sigma^{(j)}\colon f_j\,\mathrm{id}^{\mathbf{E}} \sim 0^{\mathbf{E}}\right)_1^c$ satisfying $\sigma^{(j)}\sigma^{(i)} = -\sigma^{(i)}\sigma^{(j)}$ and $\sigma^{(j)}\sigma^{(j)} = 0$.

1.7

A DG module $\mathbf{E}$ with $E_i = 0$ for $i \ll 0$ is *semi-free* if the $\mathbf{K}^\natural$-module $\mathbf{E}^\natural$ is free. This is a special case of the notion in [10, Sect. 1], where general facts on semi-freeness are discussed. For elementary proofs of the following propositions cf. [7, Sect. 1.3].

Let $\mathbf{E}$ be a semi-free DG module and $\alpha\colon \mathbf{U}' \to \mathbf{U}$ a morphism of DG modules.

1.7.1. If α is a surjective quasi-isomorphism of DG modules over $\mathbf{K}$, then each morphism $\beta\colon \mathbf{E} \to \mathbf{U}$ lifts to a morphism $\beta'\colon \mathbf{E} \to \mathbf{U}'$ such that $\beta = \alpha\beta'$; any two such liftings are homotopic by a $\mathbf{K}$-linear homotopy.

1.7.2. If α is a quasi-isomorphism then so is $\alpha \otimes_{\mathbf{K}} \mathbf{E}\colon \mathbf{U}' \otimes_{\mathbf{K}} \mathbf{E} \to \mathbf{U} \otimes_{\mathbf{K}} \mathbf{E}$.

1.7.3. If $\mathbf{E}'$ also is a semi-free DG module over $\mathbf{K}$ and if $\gamma\colon \mathbf{E}' \to \mathbf{E}$ is a quasi-isomorphism, then so is $\mathbf{V} \otimes_{\mathbf{K}} \gamma\colon \mathbf{V} \otimes_{\mathbf{K}} \mathbf{E}' \to \mathbf{V} \otimes_{\mathbf{K}} \mathbf{E}$ for each DG module $\mathbf{V}$.

Next we recall some standard constructions; details may be found in [7, Sect. 2].

1.8

Any ring homomorphism $Q \to R$ can be factored as a composition of morphisms of DG algebras $Q \hookrightarrow \mathbf{K} \twoheadrightarrow R$ with the following properties: The graded algebra $\mathbf{K}^\natural$ is the tensor product of the symmetric algebra on a free graded Q-module concentrated in non-negative even degrees with the exterior algebra on a free graded Q-module concentrated in odd degrees; the surjection $\kappa\colon \mathbf{K} \twoheadrightarrow R$ is a quasi-isomorphism. If $\mathbf{K}' \to R$ is a quasi-isomorphism of DG algebras over Q, then there is a morphism $\mathbf{K} \to \mathbf{K}'$ of DG algebras over Q inducing the identity on R.

Any R-module M becomes a DG module over $\mathbf{K}$ through the morphism κ, and there is always a quasi-isomorphism $\mathbf{E} \to M$ from a non-negatively graded semi-free DG-module $\mathbf{E}$ over $\mathbf{K}$, cf. Subsection 2.1 for a special case.

1.9

Let $\mathbf{K}$ and $\mathbf{E}$ be as in Subsection 1.8. The complex $\mathrm{Hom}_Q(\mathbf{E}, N)$ has an induced structure of DG module over the DG algebra $\Lambda = \mathbf{K} \otimes_Q k$, described by

$$\big((\xi \otimes s)(\nu)\big)(e) = (-1)^{|\nu|} s\nu(\xi e)$$

for $\xi \in \mathbf{K}$, $e \in \mathbf{E}$, $s \in k$, and $\nu\colon \mathbf{E} \to N$, a homomorphism of graded modules.

Let $\mathbf{K}'$ be a DG algebra with $\mathbf{K}'^{\natural}$ free over Q and $\mathrm{H}(\mathbf{K}') = R$. If $\phi\colon \mathbf{K} \to \mathbf{K}'$ is a morphism of DG algebras as in Subsection 1.8, then $\phi \otimes_{\mathbf{K}} \mathbf{E}\colon \mathbf{E} = \mathbf{K} \otimes_{\mathbf{K}} \mathbf{E} \to \mathbf{K}' \otimes_{\mathbf{K}} \mathbf{E}$ is a quasi-isomorphism by Proposition 1.7; clearly, $\mathbf{K}' \otimes_{\mathbf{K}} \mathbf{E}$ is a semi-free DG module over $\mathbf{K}'$.

Let $\mathbf{E}'$ be a DG module over $\mathbf{K}'$ with $\mathrm{H}(\mathbf{E}') = M$ and with $\mathbf{E}'^{\natural}$ free over Q. Proposition 1.7 yields a morphism $\beta\colon \mathbf{K}' \otimes_{\mathbf{K}} \mathbf{E} \to \mathbf{E}'$ of DG modules over $\mathbf{K}'$ that induces the identity on M. We get quasi-isomorphisms of DG modules

$$\mathrm{Hom}_Q(\mathbf{E}', N) \xrightarrow{\mathrm{Hom}_Q(\beta, N)} \mathrm{Hom}_Q(\mathbf{K}' \otimes_{\mathbf{K}} \mathbf{E}, N)$$

$$\xrightarrow{\mathrm{Hom}_Q(\phi \otimes_{\mathbf{K}} \mathbf{E}, N)} \mathrm{Hom}_Q(\mathbf{E}, N)$$

with Λ acting on the first two via the morphism $\phi \otimes_Q k\colon \Lambda = \mathbf{K} \otimes_Q k \to \mathbf{K}' \otimes_Q k$. Also, Proposition 1.7 shows that ϕ is unique up to $\mathbf{K}$-linear homotopy and that β is unique up to $\mathbf{K}'$-linear homotopy, so the compositon above is unique up to Λ-linear homotopy.

1.10

Let $\mu\colon M' \to M$ be a homomorphism of R-modules. If $\mathbf{E}'$ is a semi-free DG module over $\mathbf{K}'$ with $\mathrm{H}(\mathbf{E}') = M'$, then Proposition 1.7 yields a lifting of μ to a morphism of DG modules $\widetilde{\mu}\colon \mathbf{E}' \to \mathbf{E}$. It is unique up to homotopy, hence

$$\mathrm{Hom}_Q(\widetilde{\mu}, N)\colon\ \mathrm{Hom}_Q(\mathbf{E}, N) \to \mathrm{Hom}_Q(\mathbf{E}', N)$$

is a morphism of DG modules over Λ defined uniquely up to Λ-linear homotopy.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Choose resolutions $\mathbf{E}'$ of M' and $\mathbf{E}''$ of M'' that are semi-free DG modules over $\mathbf{K}$. There exists a differential on $\mathbf{E}^{\natural} = \mathbf{E}'^{\natural} \oplus \mathbf{E}''^{\natural}$, turning it into a DG module over $\mathbf{K}$ such that $\mathrm{H}_*(\mathbf{E}) = M$ and $0 \to \mathbf{E}' \to \mathbf{E} \to \mathbf{E}'' \to 0$ becomes an exact sequence of DG modules. It splits over Q, inducing an exact sequence of DG modules over Λ

$$0 \to \mathrm{Hom}_Q(\mathbf{E}'', N) \to \mathrm{Hom}_Q(\mathbf{E}, N) \to \mathrm{Hom}_Q(\mathbf{E}', N) \to 0.$$

In particular, the associated connecting morphism in cohomology is Λ-linear.

We are ready to establish the results stated at the beginning of this section.

Proof of Theorem 1.1. Take $\mathbf{K}$ and $\mathbf{E}$ as in Subsection 1.8. As $\mathbf{K}$ and $\mathbf{E}$ are Q-free resolutions of R and of M, respectively, we have $\operatorname{Tor}_*^Q(R,k) = \mathrm{H}_*(\mathbf{K}\otimes_Q k)$, $\operatorname{Tor}_*^Q(M,N) = \mathrm{H}_*(\mathbf{E}\otimes_Q N)$, and $\operatorname{Ext}_Q^*(M,N) = \mathrm{H}^*(\operatorname{Hom}_Q(\mathbf{E},N))$. These expressions endow $\operatorname{Tor}_*^Q(R,k)$ with a structure of graded algebra and $\operatorname{Tor}_*^Q(M,N)$, $\operatorname{Ext}_Q^*(M,N)$ with that of graded module over it. By Subsection 1.9 these structures do not depend on the choices of $\mathbf{K}$ and $\mathbf{E}$. By Subsection 1.10 they are functorial in the first argument, and connecting morphisms induced by exact sequences in that argument are Λ-linear. It is clear that $\mathrm{H}_*(\mathbf{E}\otimes_Q N)$ and $\mathrm{H}_*(\mathbf{E}\otimes_Q N)$ have the corresponding properties with respect to N.

By adjunction, $\mathrm{H}^*\operatorname{Hom}_Q\big(\mathbf{E},\operatorname{Hom}_k(N,N')\big) \cong \mathrm{H}^*\operatorname{Hom}_k(\mathbf{E}\otimes_Q N,N')$. Composed with the K'unneth map $\mathrm{H}^*\operatorname{Hom}_k(\mathbf{E}\otimes_Q N, N') \to \operatorname{Hom}_k(\mathrm{H}^*(\mathbf{E}\otimes_Q N),N')$, this isomorphism yields a canonical morphism

$$\operatorname{Ext}_Q^*(M,\operatorname{Hom}_k(N,N')) \to \operatorname{Hom}_k(\operatorname{Tor}_*^Q(M,N),N')$$

of graded modules over $\operatorname{Tor}_*^Q(R,k)$ that is bijective when N' is injective. ∎

Proof of Proposition 1.2. Recall that $\mathfrak{a}$ is a codimension c Gorenstein ideal in a local ring $(Q,\mathfrak{n},k)$ and $R = Q/\mathfrak{a}$. By [13, §1] the finite-dimensional k-algebra $\mathsf{T} = \operatorname{Tor}_*^Q(R,k)$ is self-injective. It follows that the isomorphism of T-modules $\operatorname{Ext}_Q^*(M,k) \cong \operatorname{Hom}_k(\operatorname{Tor}_*^Q(M,k),k)$ given by Theorem 1.1 transforms free direct summands into free direct summands, so $\text{f-rank}_{\mathsf{T}}\operatorname{Ext}_Q^*(M,k) = \text{f-rank}_{\mathsf{T}}\operatorname{Tor}_*^Q(M,k)$.

Choose a non-zero element $\omega\in\mathsf{T}_c$; note that it generates the socle of T and set $p = \operatorname{rank}_k(\omega\cdot\operatorname{Tor}_0^Q(M,k))$. An R-free direct summand of M yields a T-free direct summand of $\operatorname{Tor}_*^Q(M,k)$ of the same rank, hence

$$p \geq \text{f-rank}_{\mathsf{T}}\operatorname{Tor}_*^Q(M,k) \geq \text{f-rank}_R M.$$

To finish the proof we show that $\text{f-rank}_R M \geq p$.

Choose $m_1,\ldots,m_p$ in M so that $\omega\cdot(m_1\otimes 1),\ldots,\omega\cdot(m_p\otimes 1)$ is a basis of $\omega\cdot\operatorname{Tor}_0^Q(M,k)$. Let $\mu\colon R^p\to M$ be the homomorphism of Q-modules defined on the canonical basis by $\mu(e_i) = m_i$. The induced map

$$\operatorname{Tor}_*^Q(\mu,k)\colon \operatorname{Tor}_*^Q(R^p,k)\to\operatorname{Tor}_*^Q(M,k)$$

is T-linear, so $\operatorname{Tor}_c^Q(\mu,k)\big(\omega\cdot(e_i\otimes 1)\big) = \omega\cdot(m_i\otimes 1)$; hence $\operatorname{Tor}_c^Q(\mu,k)$ is injective.

Let $\mathbf{P}$ be a minimal Q-free resolution of R and $\mathbf{E}$ be a minimal Q-free resolution of M, and let $\mu'\colon \mathbf{P}^p \to \mathbf{E}$ be a morphism lifting μ. By Auslander–Buchsbaum

$$\begin{aligned}\operatorname{proj\,dim}_Q M &= \operatorname{depth} Q - \operatorname{depth}_Q M = \operatorname{depth} Q - \operatorname{depth}_Q R\\ &= \operatorname{proj\,dim}_Q R = c,\end{aligned}$$

so $E_{c+1} = 0$. Since $\mu'_c \otimes_Q k = \operatorname{Tor}_c^Q(\mu, k)$ is injective, $\mu'_c\colon (P_c)^p \to E_c$ is a monomorphism onto a direct summand, and so the R-linear map

$$\operatorname{Ext}_Q^c(\mu, Q)\colon\ \operatorname{Ext}_Q^c(M, Q) \to \operatorname{Ext}_Q^c(R^p, Q)$$

is surjective. Setting $(-)^\vee = \operatorname{Hom}_R(-, R)$ and recalling that $\operatorname{Ext}_Q^c(-, Q) \cong (-)^\vee$ as functors on the category of R-modules, we see that $\mu^\vee\colon M^\vee \to (R^p)^\vee$ is surjective and hence split.

Dualizing once more we get a split monomorphism $\mu^{\vee\vee}\colon R^p = (R^p)^{\vee\vee} \to M^{\vee\vee}$. It is the composition of $\mu\colon R^p \to M$ with the biduality map $M \to M^{\vee\vee}$, so μ is a split monomorphism, and hence $\operatorname{f-rank}_R M \geq p$, as desired. ∎

2. UNIVERSAL RESOLUTIONS

In this section $\boldsymbol{f} = \{f_1, \ldots, f_c\}$ is a *Koszul-regular* subset of Q, in the sense that $R = Q/(\boldsymbol{f}) \neq 0$) and the Koszul complex $\mathbf{K} = \mathbf{K}(\boldsymbol{f}, Q)$ satisfies $\mathrm{H}_i(\mathbf{K}) = 0$ for $i \neq 0$; equivalently, the canonical projection $\kappa\colon \mathbf{K} \to R$ is a non-zero quasi-isomorphism.

2.1

Each R-module M has a *Koszul resolution*, meaning a projective resolution of M over Q that is a DG module over $\mathbf{K}$. In view of the important role such a resolution plays in the paper, we reproduce here its simple construction.

Choose a surjection $\epsilon\colon E = \bigoplus_{i\in I(0)} Re_i(0) \to M$ and let $\mathbf{E}(0)$ be the DG module with $\mathbf{E}(0)^\natural = \bigoplus_{i\in I(0)} \mathbf{K}^\natural e_i(0)$ and $\partial\big(e_i(0)\big) = 0$. Let $\epsilon(0)\colon \mathbf{E}(0) \to M$ be the composition of ϵ with the obvious surjection $\mathbf{E}(0) \to E$. For $n \geq 0$, assume by induction that there is a DG module with $\mathbf{E}(n)^\natural = \bigoplus_{i\in I(n)} \mathbf{K}^\natural e_i(n)$ and a morphism $\epsilon(n)\colon \mathbf{E}(n) \to M$ with $\mathrm{H}_j(\epsilon(n))$ bijective for $j < n$ and surjective for $j = n$.

Let $\{z_i(n) \in \mathbf{E}(n) \mid i \in I_{n+1}\}$ be a set of cycles whose classes span $\operatorname{Ker} \mathrm{H}_n(\epsilon(n))$. Set $I(n+1) = I(n) \sqcup I_{n+1}$, and let $\mathbf{E}(n+1)$ be the DG module with $\mathbf{E}(n+1)^\natural = \bigoplus_{i\in I(n+1)} \mathbf{K}^\natural e_i(n+1)$ and differential that extends the one on $\mathbf{E}(n)$ and satisfies $\partial\big(e_i(n+1)\big) = z_i(n)$. The unique $\mathbf{K}$-linear map

$\epsilon(n+1)$: $\mathbf{E}(n+1) \to M$ that extends $\epsilon(n)$ and sends each $e_i(n+1)$ to 0 is a morphism of DG modules over $\mathbf{K}$. It is clear that $\mathrm{H}\epsilon(n+1)$ is bijective in degrees $< n+1$ and surjective in degree $n+1$. The limit map $\epsilon = \bigcup_{n\geqslant 0} \epsilon(n)$ is then a quasi-isomorphism $\mathbf{E} \to M$ and the DG module $\mathbf{E} = \bigcup_{n\geqslant 0} \mathbf{E}(n)$ is semi-free over $\mathbf{K}$.

Note that $E_n \neq 0$ for $n \geq 0$ if $\operatorname{proj\,dim}_R M = \infty$. However, if $\operatorname{proj\,dim}_Q M = q < \infty$, then setting $E'_n = 0$ for $n < q$, $E'_q = \partial(E_{q+1})$ and $E'_n = E_n$ for $n > q$ defines a subcomplex $\mathbf{E}'$ of $\mathbf{E}$, and $\mathbf{E}'' = \mathbf{E}/\mathbf{E}'$ is a projective resolution of M over Q. For degree reasons $\mathbf{E}'$ is a DG submodule of $\mathbf{E}$ over Q, whence $\mathbf{E}''$ constitutes a Koszul resolution of M over Q that has minimal possible length, equal to $\operatorname{proj\,dim}_Q M$.

2.2

The graded algebra $\Lambda = \mathrm{Tor}^Q_*(R, R)$ is computed as

$$\begin{aligned} \Lambda &= \mathrm{H}_*\big(Q\langle \xi_1, \ldots, \xi_c \mid \partial(\xi_j) = f_j\rangle \otimes_Q R\big) \\ &= \mathrm{H}_*\big(R\langle \xi_1, \ldots, \xi_c \mid \partial(\xi_j) = 0\rangle\big) = R\langle \xi_1, \ldots, \xi_c\rangle. \end{aligned}$$

The action of Λ on $\mathrm{Ext}^*_Q(M, N)$ can be computed not only from a Koszul resolution of M, but from *any* projective resolution $\mathbf{E}'$ of M over Q. In fact, if $\sigma^{(j)}$: $\mathbf{E}' \to \mathbf{E}'$ is a homotopy $f_j\,\mathrm{id}^{\mathbf{E}'} \sim 0^{\mathbf{E}'}$ and if α: $\mathbf{E}' \to N$ is a Q-linear homomorphism of cohomological degree i such that $\alpha\partial = 0$, then

$$\xi_j \operatorname{cls}(\alpha) = (-1)^i \operatorname{cls}(\alpha\sigma^{(j)}) \in \mathrm{H}^* \operatorname{Hom}_Q(\mathbf{E}', N) = \mathrm{Ext}^*_Q(M, N).$$

2.3

Let $Q[\chi_1, \ldots, \chi_c]$ be a polynomial ring on variables of cohomological degree 2. For $H = (h_1, \ldots, h_c) \in \mathbb{N}^c$, we set $\chi^H = \chi_1^{h_1} \cdots \chi_c^{h_c}$ and note that χ^H has cohomological degree $2(h_1 + \cdots + h_c)$.

The basis of $\widetilde{\mathbf{\Gamma}} = \operatorname{Hom}_Q(Q[\chi_1, \ldots, \chi_c], Q)$, dual to the basis $\{\chi^H \mid H \in \mathbb{N}^c\}$ of $Q[\chi_1, \ldots, \chi_c]$, is denoted $\{y^{(H)} \mid H \in \mathbb{N}^c\}$; accordingly, $y^{(H)}$ has homological degree $2(h_1 + \cdots + h_c)$. The induced action of $Q[\chi_1, \ldots, \chi_c]$ on $\widetilde{\mathbf{\Gamma}}$ is described by

$$\chi_j y^{(H)} = \begin{cases} y^{(h_1, \ldots, h_j - 1, \ldots, h_c)} & \text{if } h_j > 0 \\ 0 & \text{if } h_j = 0. \end{cases}$$

We set $\mathbf{\Gamma} = R \otimes_Q \widetilde{\mathbf{\Gamma}}$. Abusing notation we write $y^{(H)}$ instead of $1 \otimes_Q y^{(H)}$; the formulas above then give $\mathbf{\Gamma}$ an action of $\mathcal{S} = R \otimes_Q Q[\chi_1, \ldots, \chi_c] = R[\chi_1, \ldots, \chi_c]$.

2.4. THEOREM. *If* $\mathbf{E}$ *is a Koszul resolution of* M, *then the formula*

$$\partial(g \otimes e) = g \otimes \partial(e) + \sum_{j=1}^{c} \chi_j g \otimes \xi_j e$$

defines an R*-linear homomorphism* $\partial\colon \mathbf{\Gamma} \otimes_Q \mathbf{E}^\natural \to \mathbf{\Gamma} \otimes_Q \mathbf{E}^\natural$ *of homological degree* -1 *that satisfies* $\partial^2 = 0$. *The resulting complex* $\mathrm{G}(\mathbf{E}) = (\mathbf{\Gamma} \otimes_Q \mathbf{E}^\natural, \partial)$ *is a free resolution of* M *over* R *and a DG module over* $\mathscr{S} \otimes_Q \Lambda$ *for the obvious action.*

2.5

Shamash [31, Sect. 3] for $c = 1$ and Eisenbud [18, Sect. 7] for any c produce an R-free resolution of an R-module M out of any Q-free resolution $\mathbf{E}'$ of M over Q. They proceed in two steps, first exhibiting on $\mathbf{E}'$ a system of higher homotopies $\boldsymbol{\sigma} = \{\sigma^{(H)}\colon \mathbf{E}' \to \mathbf{E}' \mid H \in \mathbb{N}^c\}$, then using it to define on $\mathbf{E}'\{\boldsymbol{\sigma}\}^\natural = \mathbf{\Gamma} \otimes_Q \mathbf{E}'^\natural$ a differential that turns $\mathbf{E}'$ into a DG module over $\mathscr{S}$ with $\mathrm{H}_*(\mathbf{E}'\{\boldsymbol{\sigma}\}) \cong M$.

When $\mathbf{E}'$ has a Koszul structure the first step is gratuitous (cf. [18, p. 56]): just take $\sigma^{(0)}(x) = \partial(x)$, $\sigma^{(j)}(x) = \xi_j x$ for the jth unit vector (j) and $\sigma^{(H)}(x) = 0$ for $|H| > 1$. With this choice, $\mathbf{E}'\{\boldsymbol{\sigma}\} = \mathrm{G}(\mathbf{E}')$, but we give a direct construction.

The key ingredient is a specific resolution of $\mathbf{K}$ viewed as a (left) DG module over $\mathbf{K} \otimes_Q \mathbf{K}$ by means of the multiplication morphism, $a \otimes b \mapsto ab$ for $a, b \in \mathbf{K}$; this goes back to H. Cartan's "small constructions" from [15].

2.6. PROPOSITION. *Set* $\mathbf{L}^\natural = \mathbf{K}^\natural \otimes_Q \widetilde{\mathbf{\Gamma}} \otimes_Q \mathbf{K}^\natural$. *The* $\mathbf{K}^\natural \otimes_Q \mathbf{K}^\natural$*-linear homomorphism* $\partial\colon \mathbf{L}^\natural \to \mathbf{L}^\natural$ *of homological degree* -1 *given by*

$$\partial\big(y^{(H)}\big) = \sum_{j=1}^{c} \big(1 \otimes \chi_j y^{(H)} \otimes \xi_j - \xi_j \otimes \chi_j y^{(H)} \otimes 1\big)$$

turns $\mathbf{L} = (\mathbf{L}^\natural, \partial)$ *into a DG module over* $\mathbf{K} \otimes_Q \mathbf{K}$, *and the map*

$$\epsilon\colon \mathbf{L} \to \mathbf{K} \qquad \text{with} \qquad \epsilon\big(a \otimes y^{(H)} \otimes b\big) = \begin{cases} ab & \text{if } H = 0, \\ 0 & \text{if } H \neq 0 \end{cases}$$

is a quasi-isomorphism of DG modules over $\mathbf{K} \otimes_Q \mathbf{K}$.

Proof. A simple computation yields $\partial^2 = 0$. In what follows, unadorned tensor products are over Q. Clearly, ϵ is a $(\mathbf{K} \otimes \mathbf{K})$-linear chain map. To show that it is a quasi-isomorphism we use two alternative descriptions. Let $\mathbf{L}'$ be the DG module over $\mathbf{K} \otimes \mathbf{K}$ with $\mathbf{L}'^\natural = \widetilde{\mathbf{\Gamma}} \otimes \mathbf{K}^\natural \otimes \mathbf{K}^\natural$ and differential

$$\partial\big(y^{(H)}\big) = \sum_{j=1}^{c} \big(\chi_j y^{(H)}\big) \otimes \xi'_j\,, \qquad \xi'_j = 1 \otimes \xi_j - \xi_j \otimes 1.$$

For $j = 1, \ldots, c$ let $\mathbf{T}(j)$ be the complex of free Q-modules

$$\cdots \to Qy_j^{(n)}\xi'_j \xrightarrow{\partial_{2n+1}} Qy_j^{(n)} \xrightarrow{\partial_{2n}} Qy_j^{(n-1)}\xi'_j \to \cdots \to Q\xi'_j \xrightarrow{\partial_1} Qy_j^{(0)} \to 0$$

$$\partial_{2n+1}\big(y_j^{(n)}\xi'_j\big) = 0 \qquad \text{and} \qquad \partial_{2n}\big(y_j^{(n)}\big) = y_j^{(n-1)}\xi'_j \qquad \text{for} \quad n \geq 1.$$

The canonical augmentations $\epsilon(j)\colon \mathbf{T}(j) \to \mathrm{H}_0(\mathbf{T}(j)) = Q$ are clearly quasi-isomorphisms. They appear in the commutative diagram

$$\begin{tikzcd}
\big(\bigotimes_{j=1}^{c} \mathbf{T}(j)\big) \otimes \mathbf{K} \arrow[r, "\beta"] \arrow[d, "\big(\bigotimes_{j=1}^{c}\epsilon(j)\big)\otimes\mathbf{K}"'] & \mathbf{L}' \arrow[r, "\alpha"] \arrow[d, "\epsilon'"] & \mathbf{L}' \arrow[d, "\epsilon"] \\
Q^{\otimes c} \otimes \mathbf{K} \arrow[r, "\cong"] & \mathbf{K} \arrow[r, equal] & \mathbf{K}
\end{tikzcd}$$

where $\alpha(y^{(H)} \otimes a \otimes b) = a \otimes y^{(H)} \otimes b$; $\epsilon' = \epsilon\alpha$. For $H = (h_1, \ldots, h_c)$ and $i_j \in \{0, 1\}$,

$$\beta\big(y_1^{(h_1)}\xi_1'^{\,i_1} \otimes \cdots \otimes y_c^{(h_c)}\xi_c'^{\,i_c} \otimes b\big) = y^{(H)} \otimes \xi_1'^{\,i_1}\cdots\xi_c'^{\,i_c} b.$$

The map α is obviously an isomorphism.

The map β is bijective, because it is linear for the actions of $\mathbf{K}$ on the rightmost factors and maps a $\mathbf{K}^\natural$-basis of $\big(\bigotimes_{j=1}^{c} \mathbf{T}(j)^\natural\big) \otimes \mathbf{K}^\natural$ to one of $\mathbf{L}'^\natural$.

The map $\big(\bigotimes_{j=1}^{c} \epsilon(j)\big) \otimes \mathbf{K}$ is a quasi-isomorphism by the Künneth theorem.

The commutativity of the diagram implies that ϵ is a quasi-isomorphism. ∎

Proof of Theorem. 2.4. In the notation of the preceding proof, the maps

$$R \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E} \xleftarrow{\kappa\otimes_{\mathbf{K}}\mathbf{L}\otimes_{\mathbf{K}}\mathbf{E}} \mathbf{K} \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E} \cong \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E} \xrightarrow{\epsilon\otimes_{\mathbf{K}}\mathbf{E}} \mathbf{K} \otimes_{\mathbf{K}} \mathbf{E} \cong \mathbf{E}$$

are morphisms of complexes of Q-modules.

For the action via the right-hand factor $\mathbf{K}$ in $\mathbf{K} \otimes \mathbf{K} \subseteq \mathbf{L}$, the DG module $\mathbf{L}$ is semi-free over $\mathbf{K}$. Thus, ϵ is a quasi-isomorphism of semi-free $\mathbf{K}$-modules and Proposition 1.7 shows that $\epsilon \otimes_{\mathbf{K}} \mathbf{E}$ is a quasi-isomorphism.

The DG module $\mathbf{L} \otimes_{\mathbf{K}} \mathbf{E}$ over $\mathbf{K}$ is semi-free for the action via the left-hand factor $\mathbf{K}$ in $\mathbf{K} \otimes \mathbf{K} \subseteq \mathbf{L}$, because $(\mathbf{L} \otimes_{\mathbf{K}} \mathbf{E})^\natural \cong \mathbf{K}^\natural \otimes \widetilde{\mathbf{\Gamma}} \otimes \mathbf{E}^\natural$ and the Q-modules $\widetilde{\mathbf{\Gamma}}$ and $\mathbf{E}^\natural$ are free. As κ is a quasi-isomorphism, so is $\kappa \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E}$ by Proposition 1.7.

Via these quasi-isomorphisms we get $\mathrm{H}(R \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E}) \cong \mathrm{H}(\mathbf{E}) = M$. It remains to note that the isomorphism of graded modules

$$\mathbf{\Gamma} \otimes_Q \mathbf{E}^\natural \cong (R \otimes_{\mathbf{K}} \mathbf{K} \otimes \widetilde{\mathbf{\Gamma}} \otimes \mathbf{K} \otimes_{\mathbf{K}} \mathbf{E})^\natural \cong (R \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E})^\natural$$

induces an isomorphism $\mathrm{G}(\mathbf{E}) \cong R \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E}$ of DG module over $\mathcal{S} \otimes_Q \Lambda$. ∎

The situation in codimension 1 was investigated further by Eisenbud [18]. We present some of his results from the point of view of Theorem 2.4 that becomes particularly simple in this case. An alternative approach is taken by Iyengar [23, 3.2], who recovered the periodic resolution from a universal resolution based on the bar-construction.

2.7. EXAMPLE. Let f be a Q-regular element, $R = Q/(f)$, and let M be a finite R-module. Assume first that M has a Q-free resolution of the form

$$\mathbf{E} = 0 \to E_1 \xrightarrow{\delta} E_0 \to 0.$$

As $fM = 0$, we have $fE_0 \subseteq \delta(E_1)$, so there is an R-linear map $\sigma_0\colon E_0 \to E_1$ with $\delta\sigma_0 = f\,\mathrm{id}^{E_0}$. Because $\delta\sigma_0\delta = f\,\mathrm{id}^{E_0}\,\delta = \delta f\,\mathrm{id}^{E_1}$ and δ is injective, we also have $\sigma_0\delta = f\,\mathrm{id}^{E_1}$. Thus, σ_0 defines a homotopy σ from $f\,\mathrm{id}^{\mathbf{E}}$ to $0^{\mathbf{E}}$. It trivially satisfies $\sigma^2 = 0$, so $\xi x = \sigma(x)$ turns $\mathbf{E}$ into a DG module over $\mathbf{K} = Q\langle \xi \mid \delta(\xi) = f\rangle$ by Subsection 3.4. In positive degrees $2n+2$, $2n+1$, and $2n$, the resolution $\mathrm{G}(\mathbf{E})$ takes the form

$$\cdots \xrightarrow{R\otimes_Q\delta} Ry^{(n+1)} \otimes_Q E_0 \xrightarrow{R\otimes\sigma} Ry^{(n)} \otimes_Q E_1 \xrightarrow{R\otimes\delta} Ry^{(n)} \otimes_Q E_0 \xrightarrow{R\otimes\sigma} \cdots .$$

The graded algebra $R[\chi]\langle\xi\rangle$ operates on $\mathrm{G}(\mathbf{E})$ by $\chi(y^{(n)} \otimes x) = y^{(n-1)} \otimes x$ and $\xi(y^{(n)} \otimes x) = y^{(n)} \otimes \sigma(x)$.

When Q is local and $\mathbf{E}$ is a minimal Q-free resolution we have $\mathrm{Im}(R \otimes \delta) \subseteq \mathfrak{m}Ry^{(n)} \otimes_Q E_0$. Thus, $\mathrm{G}(\mathbf{E})$ is minimal precisely when $\mathrm{Im}\,\sigma \subseteq \mathfrak{n}E_1$. By exactness and periodicity we have $\mathrm{Im}(R \otimes \delta) = \mathrm{Coker}(R \otimes_Q \delta) \cong M$, so $\mathrm{G}(\mathbf{E})$ is minimal if and only if M has no free direct summand.

Keeping Q local, consider more generally an R-module M with $\mathrm{proj\,dim}_Q\, M < \infty$, and let $\mathbf{F}$ be its minimal free resolution over R. If $\mathrm{depth}\, R - \mathrm{depth}_R\, M = g$ and M' is an nth syzygy of M with $n \geq g$, then $\mathrm{depth}_R\, M' = \mathrm{depth}\, R$; if $n > g$, then in addition M' has no free direct summand, cf. [18, 7, (1.2.5)], so $\mathbf{F}_{>n} \cong \mathrm{G}(\mathbf{E}')$, where $\mathbf{E}'$ is a minimal free resolution of M' over Q.

Another source of free resolutions with Koszul structure is the following.

2.8

If $\varphi\colon Q \to M$ is a homomorphism of Q-algebras with $\varphi(\boldsymbol{f}) = 0$ and $\mathbf{E}$ is a resolution of M over Q that is a strictly commutative DG algebra, then $\mathbf{E}$ has a Koszul structure. Indeed, choose $\zeta_1, \ldots, \zeta_c \in E_1$ with $\varphi(f_j) = \partial(\zeta_j)$ for $j = 1, \ldots, c$. By the universal property of the exterior algebra φ extends to a morphism of graded Q-algebras $\mathbf{K}^\natural \to \mathbf{E}^\natural$ that maps ξ_j to ζ_j for $j = 1, \ldots, c$. This map commutes with the differentials on the generators of $\mathbf{K}$, thus defining a morphism of DG algebras $\mathbf{K} \to \mathbf{E}$ and hence a Koszul structure on $\mathbf{E}$.

Here are some instances to which this observation applies.

2.8.1. Let $M = Q/(\boldsymbol{g})$ where $\boldsymbol{g} = \{g_1, \ldots, g_d\}$ is a Q-regular set with $\boldsymbol{f} \subset (\boldsymbol{g})$, say $f_j = \sum_{i=1}^{d} a_{ij} g_i$ for $j = 1, \ldots, c$. If $\mathbf{E} = Q\langle \zeta_1, \ldots, \zeta_d \mid \partial(\zeta_i) = g_i \rangle$ is the Koszul complex resolving M over Q, then by Subsection 2.8 there is a unique morphism of DG algebras $\phi\colon \mathbf{K} \to \mathbf{E}$ with $\phi(\xi_j) = \sum_{i=1}^{d} a_{ij} \zeta_i$.

In this special case $\mathrm{G}(\mathbf{E})$ is Tate's resolution of M over R, denoted

$$R\left\langle \zeta_1, \ldots, \zeta_d; y_1, \ldots, y_c \,\middle|\, \partial(\zeta_i) = \overline{g}_i\, ;\, \partial(y_j) = \sum_{i=1}^{d} \overline{a}_{ij} \zeta_i \right\rangle$$

in [33, Theorem 4]. The proof of Theorem 2.4 does not use Tate's result; one may as well reverse the roles, as was done in [7, (9.1.1)].

2.8.2. With $\boldsymbol{f}$ and $\boldsymbol{g}$ as above, assume that $\boldsymbol{f} \subset (\boldsymbol{g})^h$ for some $h \geq 1$. Buchsbaum and Eisenbud [14, (3.2)] construct a free resolution $\mathbf{E}$ of $M = Q/(\boldsymbol{g})^h$ over Q with $\partial(\mathbf{E}) \subseteq (\boldsymbol{g})\mathbf{E}$. The rank of E_i, computed from [14, (2.5.c)], is equal to

$$a_i(h, d) = \binom{h + d - 1}{h + i - 1} \binom{h + i - 2}{h - 1}.$$

Srinivasan [32, (3.4), (3.6)] puts on $\mathbf{E}$ a DG algebra structure with $\mathbf{E}_{\geqslant 1} \cdot \mathbf{E}_{\geqslant 1} \subseteq (\boldsymbol{g})\mathbf{E}$.

2.8.3. Buchsbaum and Eisenbud [13, (1.3)] show that if $E_0 = Q$ and $E_i = 0$ for $i > 3$, then $\mathbf{E}$ has a structure of DG algebra.

We finish this section with a further remark on multiplicative structures.

2.9

Let $\boldsymbol{\Gamma}$ be the graded R-module from Subsection 2.3. The multiplication table

$$y^{(H)} y^{(H')} = \binom{H + H'}{H} y^{(H+H')} \qquad \text{for} \quad H, H' \in \mathbb{N}^c$$

turns $\boldsymbol{\Gamma}$ into a graded R-algebra, known as the *free divided powers algebra* over R in the Γ-variables $y_1, \ldots, y_c$. If $\mathbf{E}$ is a DG algebra over $\mathbf{K}$, then $\mathrm{G}(\mathbf{E})$ becomes a DG algebra with underlying graded algebra $\boldsymbol{\Gamma} \otimes_Q \mathbf{E}^{\natural}$, and the inclusion $\mathbf{E} = \mathbf{E} \otimes_Q 1 \subseteq \mathrm{G}(\mathbf{E})$ is one of DG algebras. In particular, the DG module $\mathbf{L}$ of Proposition 2.6 is a DG algebra that contains $\mathbf{K} \otimes_Q \mathbf{K}$ as a DG subalgebra.

In the parlance of Quillen's homotopical algebra [27], transposed by him into commutative algebra in [28], the DG algebra maps $\mathbf{K} \otimes \mathbf{K} \to \mathbf{L} \to \mathbf{K}$ constitute a *model* of the multiplication map $\mathbf{K} \otimes \mathbf{K} \to \mathbf{K}$ and $\mathscr{S} \cong \mathrm{H}(\mathrm{Hom}_{\mathbf{K} \otimes \mathbf{K}}(\mathbf{L}, R))$ represents the (*derived* or *hyper*-) *Hochschild cohomology* of the algebra $Q \to R$.

3. COHOMOLOGY OPERATORS

In this section $\boldsymbol{f} = \{f_1, \ldots, f_c\}$ is a Koszul-regular set in a commutative ring Q, $\mathbf{K}$ is the Koszul complex on $\boldsymbol{f}$, and $R = Q/(\boldsymbol{f})$.

Gulliksen [21] proves that Ext and Tor functors over R are naturally graded modules over a polynomial ring with variables of cohomological degree 2. Eisenbud [18] introduces another action and the two structures agree up to sign; cf. [11].

We give a new construction of Eisenbud's operators. It is based on resolutions of R-modules that are DG modules over the graded algebra $\mathcal{S} \otimes_Q \Lambda$, where the *ring of cohomology* operators $\mathcal{S}$ is the polynomial ring $R[\chi_1, \ldots, \chi_c]$ with variables χ_j of cohomological degree 2 and the *algebra of homology operators* Λ is the exterior algebra $R\langle \xi_1, \ldots, \xi_c\rangle$ with variables ξ_j of cohomological degree -1.

3.1. THEOREM. *For R-modules M, N the algebra $\mathcal{S}$ acts on* $\operatorname{Ext}_R^*(M, N)$ *through*

$$\chi_j\colon \operatorname{Ext}_R^n(M, N) \to \operatorname{Ext}_R^{n+2}(M, N) \qquad \textit{for } n \in \mathbb{Z} \textit{ and } j = 1, \ldots, c.$$

The action is natural in both M and N.

Exact sequences of R-modules induce $\mathcal{S}$-linear connecting morphisms.

The main work is embedded in the following constructions.

3.2

To remind the reader that the variables χ_j have cohomological degree 2 we use $\{\ \}$ to denote shifts on graded $\mathcal{S}$-modules. For instance, multiplication with a linear form in $\chi_1, \ldots, \chi_c$ defines a morphism from $\mathcal{S}$ to $\mathcal{S}\{2\}$.

For a Koszul resolution $\mathbf{E}$ of M let $\mathrm{C}(\mathbf{E}, N)$ be the DG module over $\mathcal{S}$, with

$$\mathrm{C}(\mathbf{E}, N)^\natural = \bigoplus_{u=0}^{\infty} \mathcal{S} \otimes_R \operatorname{Hom}_Q(E_u, N)\{-2u\}$$

and canonical action of $\mathcal{S}$, whose differential acts by the formula

$$\partial(\chi^H \otimes \gamma) = (-1)^{|\gamma|}\Big(\chi^H \otimes \gamma \circ \partial^{\mathbf{E}} + \sum_{j=1}^{c} \chi_j \chi^H \otimes \gamma \circ \xi_j\Big).$$

One sees by direct comparison that $\mathrm{C}(\mathbf{E}, N) = \operatorname{Hom}_R(\mathrm{G}(\mathbf{E}), N)$ as complexes of R-modules, where $\mathrm{G}(\mathbf{E})$ is the R-free resolution of M provided by Theorem 2.4.

We bigrade the R-module underlying $\mathrm{C}(\mathbf{E}, N)$ by assigning to the elements of $\mathcal{S}^v \otimes_R \mathrm{Hom}(E_u, N)\{-2u\}$ *complex* degree $-u$, *operator* degree $2u+2v$, and *bi*degree $(-u, 2u+2v)$; the sum of the complex and operator degrees of a bihomogeneous element is its *cohomological* degree, here equal to $u+2v$.

The action of $\chi_j \in \mathcal{S}$ has bidegree $(0,2)$ and cohomological degree 2.

The action of $\xi_j \in \Lambda$ has bidegree $(1,-2)$ and cohomological degree -1

The first component of ∂ has bidegree $(-1,2)$ and cohomological degree 1.

The second component of ∂ has bidegree $(1,0)$ and cohomological degree 1.

Thus, $\mathrm{C}(\mathbf{E}, N)$ is a DG module over the algebra $\mathcal{S} \otimes_R \Lambda$ with polynomial variables χ_j of bidegree $(0,2)$ and exterior variables ξ_j of bidegree $(-1,2)$. As in Subsection 1.9, if $\mathbf{E}'$ is any Koszul resolution of M, then $\mathrm{C}(\mathbf{E}', N)$ and $\mathrm{C}(\mathbf{E}, N)$ are linked by quasi-isomorphisms of DG modules over $\mathcal{S} \otimes_R \Lambda$ defined uniquely up to homotopy.

3.3

Let $\mu\colon M' \to M$ be a homomorphism of R-modules. If $\mathbf{E}'$ is a Koszul resolution of M' and if $\widetilde{\mu}\colon \mathbf{E}' \to \mathbf{E}$ is a lifting of μ to a morphism of DG modules over $\mathbf{K}$, then $\mathcal{S} \otimes_Q \widetilde{\mu}\colon \mathrm{G}(\mathbf{E}') \to \mathrm{G}(\mathbf{E})$ is a lifting of μ to a morphism of DG modules over $\mathcal{S} \otimes_R \Lambda$. If $\widetilde{\mu}'\colon \mathbf{E}' \to \mathbf{E}$ also lifts μ, then Proposition 1.7 provides a $\mathbf{K}$-linear homotopy $\tau\colon \widetilde{\mu} \sim \widetilde{\mu}'$, so $\mathcal{S} \otimes_Q \tau$ is an $\mathcal{S} \otimes_R \Lambda$-linear homotopy $\mathcal{S} \otimes_Q \widetilde{\mu} \sim \mathcal{S} \otimes_Q \widetilde{\mu}'$.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. By Subsection 1.10 it is the homology sequence of an exact sequence $0 \to \mathbf{E}' \to \mathbf{E} \to \mathbf{E}'' \to 0$ of DG modules over $\mathbf{K}$ that splits over Q. The latter sequence gives rise to the exact sequence of DG modules over $\mathcal{S} \otimes_R \Lambda$

$$0 \to \mathrm{C}(\mathbf{E}'', N) \to \mathrm{C}(\mathbf{E}, N) \to \mathrm{C}(\mathbf{E}', N) \to 0.$$

Proof of Theorem 3.1. Let $\mathcal{S}$ act on $\mathrm{Ext}_R^*(M, N)$ through the isomorphism

$$\mathrm{Ext}_R^*(M, N) \cong \mathrm{H}^*(\mathrm{C}(\mathbf{E}, N))$$

that results from Subsection 3.2. The discussion in Subsection 3.3 shows that this structure is independent of the choice of $\mathbf{E}$, that it is functorial in M, and that exact sequences in the first argument induce connecting morphisms of $\mathcal{S}$-modules.

The corresponding properties in the second argument are clear. ∎

3.4

As $\operatorname{Ext}_R^*(M, N)$ is the cohomology of a DG module over $\mathcal{S} \otimes_Q \Lambda$, it also inherits a structure of graded module over the algebra Λ of homology operators.

However, this action is trivial: $\xi_j \operatorname{Ext}_R^*(M, N) = 0$ for $j = 1, \dots, c$.

Indeed, let $\mathbf{L}$ be the DG module over $\mathbf{K} \otimes_Q \mathbf{K}$ defined in Proposition 2.6. Consider map $\tau'_j\colon \mathbf{L} \to \mathbf{L}$ of degree 2 given by

$$\tau'_j\big(a \otimes y^{(h_1, \dots, h_j, \dots, h_c)} \otimes b\big) = (h_j + 1) a \otimes y^{(h_1, \dots, h_j + 1, \dots, h_c)} \otimes b.$$

A straightforward computation then shows that τ'_j is a $\mathbf{K} \otimes_Q \mathbf{K}$-linear homotopy between $\xi'_j \operatorname{id}^{\mathbf{L}}$ and 0. It follows that it induces a homotopy between the action of ξ_j and the zero map on $\mathbf{E}\{\xi\} = R \otimes_{\mathbf{K}} \mathbf{L} \otimes_{\mathbf{K}} \mathbf{E}$ and hence between the action of ξ_j and the zero map on $\mathrm{C}(\mathbf{E}, N) = \operatorname{Hom}_R(\mathbf{E}\{\xi\}, N)$. The desired assertion follows.

3.5

Theorem 3.1 gives rise to some useful canonical morphisms.

3.5.1. The homomorphism $Q \to R$ induces a *change-of-rings morphism*

$$\rho_{MN}\colon \operatorname{Ext}_R^*(M, N) \to \operatorname{Ext}_Q^*(M, N).$$

This is the map induced in homology by the morphism of complexes of R-modules

$$\mathrm{C}(\mathbf{E}, N) \to \frac{\mathrm{C}(\mathbf{E}, N)}{(\chi_1, \dots, \chi_c)\mathrm{C}(\mathbf{E}, N)} \cong \operatorname{Hom}_Q(\mathbf{E}, N).$$

It induces a decomposition of ρ_{MN} of the form

$$\operatorname{Ext}_R^*(M, N) \to \frac{\operatorname{Ext}_R^*(M, N)}{(\chi_1, \dots, \chi_c) \operatorname{Ext}_R^*(M, N)} \xrightarrow{\overline{\rho}_{MN}} \operatorname{Ext}_Q^*(M, N)$$

in which the first map is the canonical projection; we refer to the second map as the *reduced change-of-rings morphism*. By construction, ρ_{MN} is a morphism of graded Λ-modules, so the preceding Remark 3.4 implies that

$$\begin{aligned} \operatorname{Im}(\rho_{MN}) &\subseteq \operatorname{ann}_{\operatorname{Ext}_Q^*(M,N)}(\xi_1, \dots, \xi_c) \\ &= \{\varepsilon \in \operatorname{Ext}_Q^*(M, N) \mid \xi_j \varepsilon = 0 \text{ for } 1 \le j \le c\}. \end{aligned}$$

3.5.2. The action of $\mathcal{S}$ on $\operatorname{Ext}_R^0(M, N)$ is described by the morphism

$$\varkappa_{MN}\colon \mathcal{S} \otimes_R \operatorname{Ext}_R^0(M, N) \to \operatorname{Ext}_R^{\text{even}}(M, N)$$

of graded $\mathcal{S}$-modules induced by the embedding of complexes

$$\mathcal{S} \otimes_R \operatorname{Ext}_R^0(M, N) = \mathcal{S} \otimes_R \operatorname{Hom}_Q(M, N) \subseteq \mathcal{S} \otimes_R \operatorname{Hom}_Q(\mathbf{E}, N).$$

3.6

Let $\mathbf{F}$ be an R-projective resolution of M and $M' = \partial(F_s)$.

The iterated connecting morphism ∂: $\mathrm{Ext}^*_R(M', N)\{-s\} \to \mathrm{Ext}^*_R(M, N)$ is surjective in degree s and bijective in degrees $> s$. When R is local, M is finite, and $\mathbf{F}$ is its minimal resolution, the homomorphism ∂ is bijective in degrees $\geq s$. As ∂ is $\mathcal{S}$-linear, the $\mathcal{S}$-module Ext can be studied by shifting degrees.

In this connection it is useful to note the following: Due to the isomorphisms $\mathrm{Ext}^i_Q(F_j, R) \cong \bigwedge^i(R^c) \otimes_R \mathrm{Hom}_R(F_j, R)$, if $\mathrm{Ext}^i_Q(M, N) = 0$ for $i > g$ then $\mathrm{Ext}^i_Q(M', N) = 0$ for $i > \max\{c, g - s\}$ by the cohomology exact sequence.

3.7

Assume that M has a Koszul resolution $\mathbf{E}$ over Q such that $\mathrm{Hom}_Q(\partial^{\mathbf{E}}, N) = 0$.

In this case the DG module $\mathrm{C}(\mathbf{E}, N)$ simplifies to a *complex* $\mathscr{C}^\bullet(M, N)$ of graded $\mathcal{S}$-modules with $\mathscr{C}^{-p}(M, N) = \mathcal{S} \otimes_R \mathrm{Ext}^{-p}_Q(M, N)\{2p\}$ and whose differential d is given by multiplication with $\sum_{j=1}^c \chi_j \otimes_R \xi_j$; note that d has complex degree 1, operator degree 0, and cohomological degree 1. Accordingly, $\mathrm{Ext}^*_R(M, N)$ acquires a canonical decomposition as a direct sum of graded $\mathcal{S}$-modules

$$\mathrm{Ext}^{\mathrm{even}}_R(M, N) = \bigoplus_{p \text{ even}} \mathscr{H}^{-p}(\mathscr{C}^\bullet(M, N))\{p\},$$

$$\mathrm{Ext}^{\mathrm{odd}}_R(M, N) = \bigoplus_{p \text{ odd}} \mathscr{H}^{-p}(\mathscr{C}^\bullet(M, N))\{p\}.$$

This computation applies, in particular, to the case when (Q, m, k) is a local ring, M has a *minimal* free Koszul resolution $\mathbf{E}$ over Q, and $N = k$.

3.8. Example. Let $P = k[x_1, \ldots, x_d]$ be a polynomial ring over a commutative ring k and let $\boldsymbol{h} = h_1, \ldots, h_c$ be a Koszul-regular sequence in P such that $R = P/(\boldsymbol{h})$ is flat over k. In this context, overbars denote images in R.

The kernel of the multiplication map $P \otimes_k P \to P$ is generated by the regular sequence $\mathbf{d}\boldsymbol{x} = 1 \otimes x_1 - x_1 \otimes 1, \ldots, 1 \otimes x_d - x_d \otimes 1$. Set $Q = P \otimes_k R$ and let $\boldsymbol{g} = g_1, \ldots, g_c$ be the image of $\mathbf{d}\boldsymbol{x}$ in Q, so that $g_i = 1 \otimes \overline{x}_i - x_i \otimes 1$. Clearly,

$$Q/(\boldsymbol{g}) \cong P \otimes_{P \otimes_k P} (P \otimes_k R) \cong R.$$

The isomorphism $\mathbf{K}(\boldsymbol{g}; Q) \cong \mathbf{K}(\mathbf{d}\boldsymbol{x}; P \otimes_k P) \otimes_{P \otimes_k P} Q$ yields

$$\mathrm{H}_*(\mathbf{K}(\boldsymbol{g}; Q)) \cong \mathrm{Tor}^{P \otimes_k P}_*(P, Q).$$

By assumption, the Koszul complex $\mathbf{K}(\boldsymbol{h};P)$ is a free resolution of R over P. This implies that $P\otimes_k \mathbf{K}(\boldsymbol{h};P)$ is a free resolution of Q over $P\otimes_k P$, so

$$\operatorname{Tor}_*^{P\otimes_k P}(P,Q)\cong \mathrm{H}_*\left(P\otimes_{P\otimes_k P}(P\otimes_k \mathbf{K}(\boldsymbol{h};P))\right)\cong \mathrm{H}_*(\mathbf{K}(\boldsymbol{h};P))\cong R.$$

These isomorphisms show that $\mathbf{K}=\mathbf{K}(\boldsymbol{g};Q)$ is a resolution of R over Q.

Setting $\boldsymbol{f}=f_1,\dots,f_c$ with $f_j=h_j\otimes 1$, we see that the complex $\mathbf{E}=\mathbf{K}(\boldsymbol{f};Q)$ is isomorphic to $\mathbf{K}(\boldsymbol{h};P)\otimes_k R$ and hence resolves $R\otimes_k R$ over Q, as R is k-flat.

Since $Q/(\boldsymbol{g})\cong R$, we have $\boldsymbol{f}\subseteq(\boldsymbol{g})$; hence $\operatorname{Hom}_Q(\partial^{\mathbf{E}},N)=0$ for each R-module N. Now Subsection 3.7 yields a direct sum decomposition

$$\operatorname{Ext}^n_{R\otimes_k R}(R,N)\cong \bigoplus_{-p+q=n}\mathscr{H}^{-p}(\mathscr{C}^\bullet(M,N))^q$$

along with a similar decomposition for $\operatorname{Tor}_n^{R\otimes_k R}(R,N)$. These Tor's represent *Hochschild homology* as R is k-flat; the preceding Ext's represent Hochschild cohomology if R is furthermore projective as k-module.

If k contains $\mathbb{Q}$, then the decompositions above coincide with the *Hodge* (or λ-) *decompositions* of Hochschild (co-)homology; cf. [26, 4.5.10]. The existence of such a decomposition for complete intersections of arbitrary characteristic was apparently first recorded by J. A. Guccione and J. J. Guccione [20]. Furthermore, Subsection 2.8.1 provides an explicit resolution of $R=Q/(\boldsymbol{g})$ over $R\otimes_k R$: it is the one constructed by Wolffhardt [34, Theorem 2] through a direct computation.

3.9. Example. Let $(Q,\mathfrak{n},k)$ be a local ring; let $\boldsymbol{g}=g_1,\dots,g_d$ and $\boldsymbol{f}=f_1,\dots,f_c$ be Q-regular sequences with $\boldsymbol{f}\subseteq(\boldsymbol{g})^h$ for some $h\geq 1$. Set $R=Q/(\boldsymbol{f})$ and $M=Q/(\boldsymbol{g})^h$. Changing the generators of $\boldsymbol{f}$ if necessary, we may further assume that $f_1,\dots,f_b$ project to a k-linearly independent subset of $(\boldsymbol{g})^h/\mathfrak{n}(\boldsymbol{g})^h$, while $f_{b+1},\dots,f_c$ are in $\mathfrak{n}(\boldsymbol{g})^h$. Resolve R over Q by the Koszul complex $\mathbf{K}$ on $\boldsymbol{f}$.

3.9.1. If $h=1$, changing if necessary the generators of $\boldsymbol{g}$, we may assume that $g_j=f_j$ for $1\leq j\leq b$. Resolve M by the Koszul complex $\mathbf{E}$ on $\boldsymbol{g}$. The morphism of DG algebras $\phi\colon \mathbf{K}\to\mathbf{E}$ from Subsection 2.8 induces a homomorphism of algebras $\overline{\phi}\colon k\langle\xi_1,\dots,\xi_c\rangle\to k\langle\zeta_1,\dots,\zeta_d\rangle$ with $\overline{\phi}(\xi_j)=\overline{\zeta}_j$ for $j\leq b$ and $\overline{\phi}(\xi_j)=0$ for $j>b$. Thus, we get an isomorphism of complexes of graded $\mathscr{R}$-modules

$$\mathscr{C}^\bullet(M,k)\cong\mathbf{K}(\chi_1,\dots,\chi_b,0,\dots,0;\mathscr{R})$$

with $\mathscr{C}^\bullet(M,k)$ as in Subsection 3.7. It yields an isomorphism of graded $\mathscr{R}$-modules

$$\operatorname{Ext}^*_R(M,k)\cong\mathscr{A}\otimes_k\textstyle\bigwedge^*\left(k^{d-b}\{-1\}\right)$$

where the graded $\mathcal{R}$-algebra $\mathcal{A}$ is defined by

$$\mathcal{A} = \frac{\mathcal{R}}{(\chi_1, \dots, \chi_b)} \cong k[\chi_{b+1}, \dots, \chi_c].$$

3.9.2. If $h \geq 2$, let $\mathbf{E}$ be the minimal DG algebra resolution of M from Subsection 2.8, and let $\phi\colon \mathbf{K} \to \mathbf{E}$ be a morphism of DG algebras that lifts the canonical projection $R = Q/(f) \to Q/(\boldsymbol{g})^h = M$. Since $\mathbf{E}_{\geqslant 1} \cdot \mathbf{E}_{\geqslant 1} \subseteq \mathfrak{n}\mathbf{E}$, we get

$$\operatorname{Tor}^Q_*(M, k) = \frac{\mathsf{T}}{(\xi_{b+1}, \dots, \xi_c)\mathsf{T} + \mathsf{T}_{\geqslant 2}} \oplus k^{a_1(h,d)-b}[-1] \bigoplus_{i=2}^{d} k^{a_i(h,d)}[-i]$$

as DG modules over $\mathsf{T} = k\langle \xi_1, \dots, \xi_c\rangle$. From Theorem 1.1 we obtain an isomorphism $\operatorname{Ext}^*_Q(M, k) \cong \operatorname{Hom}_k(\operatorname{Tor}^Q_*(M, k), k)$ of graded T-modules, and so Subsection 3.7 shows that $\operatorname{Ext}^*_R(M, k)$ can be computed from a complex $\mathrm{C}(M, k)$ with $\mathscr{C}^i(M, k) = \mathcal{R}^{a_i(h,d)}\{-2i\}$ for $i \geq 0$ and with only non-zero differential

$$\mathcal{R}^{a_1(h,d)}\{-2\} \xrightarrow{(\chi_1 \cdots \chi_b \ 0 \cdots 0)} \mathcal{R}.$$

Thus, we get an isomorphism of graded $\mathcal{R}$-modules

$$\operatorname{Ext}^*_R(M, k) \cong \mathcal{A} \oplus \mathcal{B}\{1\} \oplus \mathcal{R}^{a_1(h,d)-b}\{-1\} \bigoplus_{i=2}^{d} \mathcal{R}^{a_i(h,d)}\{-i\}$$

where $a_i(h, d) = \binom{h+d-1}{h+i-1}\binom{h+i-2}{h-1}$ as in Subsection 2.8.2, $\mathcal{A}$ is the $\mathcal{R}$-algebra defined in Subsection 3.9.1, and

$$\mathcal{B} = \operatorname{Ker}\left(\mathcal{R}^{\mathscr{C}}\{-2\} \xrightarrow{(\chi_1 \cdots \chi_b)} \mathcal{R}\right).$$

3.10

We recall the definitions of some invariants attached to a local ring $(R, \mathfrak{m}, k)$.

The *embedding dimension* of R is the minimal number of generators of $\mathfrak{m}$, denoted $\operatorname{edim} R$. If $\operatorname{edim} R = d$, then $\operatorname{codim} R = d - \dim R$ is the *codimension* of R. We define the *order* of R by setting $\operatorname{ord} R = 1$ if R is regular and $\operatorname{ord} R = \inf\{n \in \mathbb{N} \mid \operatorname{length}_R(R/\mathfrak{m}^{n+1}) < \binom{d+n}{n}\}$ otherwise.

For any minimal Cohen presentation of the completion $\widehat{R}$ of R as a residue $Q/\mathfrak{a}$ of a regular local ring $(Q, \mathfrak{n}, k)$ by an ideal $\mathfrak{a} \subseteq \mathfrak{n}^2$ one has $\operatorname{edim} R = \dim Q$, $\operatorname{codim} R = \operatorname{height} \mathfrak{a}$, and $\operatorname{ord} R = \sup\{n \in \mathbb{N} \mid \mathfrak{a} \subseteq \mathfrak{n}^n\}$ unless R is regular.

A local ring R is a *complete intersection* if the defining ideal in some (or, equivalently, in any) Cohen presentation $\widehat{R} = Q/\mathfrak{a}$ can be generated by a regular sequence. We can determine the $\mathcal{R}$-modules $\operatorname{Ext}^*_{\widehat{R}}(R/\mathfrak{m}^n, k)$ for the initial values of n by applying Subsection 3.9 to a regular sequence $\boldsymbol{g}$ of generators of the maximal ideal of Q. The numerical expressions of these computations are formulas for Poincaré series recorded in the following corollary. For $n = 1$ the result goes back to Tate [33, Theorem 6]; for $n = 2$ it is proved in [6, (2.1)] by a more complicated argument.

3.11. Corollary. *Let $(R, \mathfrak{m}, k)$ be a local complete intersection with embedding dimension d, codimension c, and order h. If* $\operatorname{length}_R(R/\mathfrak{m}^{h+1}) = \binom{d+h}{h} - b$, *then*

$$\sum_{j=0}^{\infty} \beta_j^R(R/\mathfrak{m}^n)t^j$$
$$= \begin{cases} \dfrac{\sum_{i=0}^{d} a_i(n,d)t^i}{(1-t^2)^c} & \textit{if } n < h, \\[2ex] \dfrac{\sum_{i=1}^{d} a_i(n,d)t^i}{(1-t^2)^c} + \dfrac{(1+t)(1-t^2)^b - 1}{t(1-t^2)^c} & \textit{if } n = h. \end{cases}$$

The complex $\mathscr{C}^\bullet(M, k)$ appearing in Subsection 3.7 is closely related to the construction introduced by J. Bernstein et al. in their study [12] of the derived category of the category of coherent sheaves on projective space; cf. the exposition of S. Gelfand [19] for details. That theory is not used here, so we only sketch the connection.

3.12

Let $\mathsf{T} = k\langle \xi_1, \ldots, \xi_c\rangle$ be the exterior algebra on variables of homological degree 1 and $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$ be the polynomial algebra on variables of cohomological degree 2. We identify the k-vector spaces T_1 and $\operatorname{Hom}_k(\mathcal{R}^2, k)$ by proclaiming $\xi_1, \ldots, \xi_c$ and $\chi_1, \ldots, \chi_c$ dual bases.

To a graded T-module M one associates the complex of graded $\mathcal{R}$-modules

$$\mathscr{M}^\bullet = \cdots \to \mathcal{R} \otimes_k \mathsf{M}^p\{-2p\} \xrightarrow{\partial} \mathcal{R} \otimes_k \mathsf{M}^{p-1}\{-2p+2\} \to \cdots$$

whose differential is given by multiplication with $\sum_{j=1}^{c} \chi_j \otimes \xi_j$. The complex $\mathscr{M}^\bullet$ is *linear*, in the sense that in terms of bases of the modules $\mathscr{M}^p$ the differential is given by matrices of linear forms in $\chi_1, \ldots, \chi_c$; it is also *standard*, meaning that each $\mathscr{M}^p$ is generated in degree $2p$.

In the opposite direction, for each standard linear complex of graded $\mathcal{R}$-modules

$$\mathscr{N}^\bullet = \cdots \to \mathscr{N}^{p+1} \xrightarrow{\partial} \mathscr{N}^p \xrightarrow{\partial} \mathscr{N}^{p-1} \to \cdots$$

let N be the graded vector space with $\mathsf{N}^p = \mathscr{N}^p \otimes_{\mathscr{R}} k$. The differential ∂ induces k-linear maps $\mathsf{N}^p \to \mathscr{R}^2 \otimes_k \mathsf{N}^{p-1}$ for $p \in \mathbb{Z}$. Under the canonical isomorphisms

$$\begin{aligned}\operatorname{Hom}_k(\mathsf{N}^p, \mathscr{R}^2 \otimes_k \mathsf{N}^{p-1}) &\cong \operatorname{Hom}_k(\operatorname{Hom}_k(\mathscr{R}^2, k) \otimes_k \mathsf{N}^p, \mathsf{N}^{p-1}) \\ &= \operatorname{Hom}_k(\mathsf{T}_1 \otimes_k \mathsf{N}^p, \mathsf{N}^{p-1})\end{aligned}$$

these maps define k-linear homomorphisms $\mathsf{T}_1 \otimes_k \mathsf{N}^p \to \mathsf{N}^{p-1}$. The equality $\partial^2 = 0$ guarantees that they produce on N a structure of graded T-module.

The assignments $\mathsf{M} \mapsto \mathscr{M}^\bullet$ and $\mathscr{N}^\bullet \mapsto \mathsf{N}$ are quasi-inverse additive equivalences between the category of graded T-modules and the category of standard linear complexes of graded $\mathscr{R}$-modules. This equivalence is described by Yoshino in [35]. The functors themselves are restrictions of functors between the categories of complexes of graded T-modules and complexes of graded $\mathscr{S}$-modules; the equivalence above may be viewed as a specialization of the BGG correspondence [12, 19].

4. CODIMENSION TWO

This section is devoted to the proof of the following result.

4.1. THEOREM. *Let $R = Q/(f_1, f_2)$ where f_1, f_2 is a regular sequence in a local ring $(Q, \mathfrak{n}, k)$, and let $\mathscr{S} = R[\chi_1, \chi_2]$ be the corresponding ring of cohomology operators. Let M be a finite R-module, and set* $\operatorname{depth} R - \operatorname{depth}_R M = g$.

If M has finite projective dimension over Q then $\operatorname{Ext}^*_R(M, k)$ is generated over the ring $\mathscr{R} = \mathscr{S} \otimes_R k = k[\chi_1, \chi_2]$ by elements of degree $\leq m$, where

$$m = \max\left\{2\beta^R_g(M)\,,\ 2\beta^R_{g+1}(M) + 1\right\} + g + 1.$$

This is an *effective* version of a general finiteness result of Gulliksen [21]; cf. Corollary 6.2 below. Our proof establishes much more than the statement above: in Subsection 4.7 we obtain $\operatorname{Ext}^{\geqslant m}_R(M, k)$ as a direct sum of explicitly given indecomposable graded $\mathscr{R}$-modules.

We do not know whether the bound on the degrees of the generators is sharp. By applying an argument from the proof of [9, (7.4)] to an example of [18, p. 44] we see that it cannot be cut more than in half:

4.2. EXAMPLE. Let $R = k[[x_1, x_2]]/(f_1, f_2)$, so that $g = 0$ for each finite R-module. Tate's minimal free resolution $\mathbf{G}$ of k over R (cf. Subsection 2.8). has $\operatorname{rank}_R G_n = n + 1$ for $n \geq 0$. Since R is self-injective, dualization yields an exact sequence

$$0 \to k \to \operatorname{Hom}_R(G_0, R) \to \cdots \to \operatorname{Hom}_R(G_{s-1}, R) \xrightarrow{\partial_{-s}} \operatorname{Hom}_R(G_s, R) \to \cdots$$

with $M_s = \mathrm{Im}(\partial_{-s}) \subseteq m\,\mathrm{Hom}_R(G_s, R)$. The R-module M_s has $\beta_0^R(M_s) = s$ and $\beta_1^R(M_s) = s - 1$; by the theorem, $\mathrm{Ext}_R^*(M_s, k)$ is generated in degree $\leq 2s + 1$.

Since k is an sth syzygy of M_s, Subsection 3.6 yields an exact sequence

$$0 \to \mathrm{Ext}_R^*(k, k)[-s] \to \mathrm{Ext}_R^*(M_s, k) \to \mathcal{N} \to 0$$

of graded $\mathcal{R}$-modules, with $\mathrm{rank}_k \mathcal{N} < \infty$. By Subsection 3.9 the $\mathcal{R}$-module $\mathrm{Ext}_R^*(k, k)$ is free, with one basis element apiece in degrees 0 and 2 and two basis elements in degree 1. It follows that $\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{N}, \mathrm{Ext}_R^*(k, k)) = 0$, so the sequence splits, showing that $\mathrm{Ext}_R^*(M_s, k)$ has a *minimal* generator in degree $s + 2$.

We now take a careful look at graded modules over the algebra $\mathrm{Tor}^Q(R, k)$.

4.3

Let $\mathsf{T} = k\langle \xi_1, \xi_2 \rangle$ be the exterior algebra on variables $\xi_1, \xi_2 \in \mathsf{T}_1 = \mathsf{T}^{-1}$.

4.3.1. As the algebra T is self-injective $\mathsf{K} = \mathrm{Hom}_k(\mathsf{T}, k)$ is a free T-module on a basis element contained in $\mathsf{K}_{-2} = \mathsf{K}^2$. Each graded free T-module is isomorphic to a direct sum of translates of K. A T-module N has a no free direct summand if and only if $\xi_1\xi_2\mathsf{N} = 0$, that is, if and only if it is a module over the algebra $\mathsf{T}/(\xi_1\xi_2)$.

A straightforward way to produce $\mathsf{T}/(\xi_1\xi_2)$-modules is to fix an integer p, set $\mathsf{N}^i = 0$ for $i \neq p, p-1$, choose finite dimensional k-vector spaces N^p and N^{p-1} and define the action of $\mathsf{T}/(\xi_1\xi_2)$ by specifying linear maps $\xi_1, \xi_2 \colon \mathsf{N}^p \to \mathsf{N}^{p-1}$. Any such choice yields a module structure. Here are some simple examples.

4.3.2. For each integer $n \geq 0$ let $\mathsf{L}(n)$ be the T-module with non-zero components $\mathsf{L}(n)^1 = k^n$ and $\mathsf{L}(n)^0 = k^{n+1}$ and action

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For each integer $n > 0$ let $\mathsf{L}(n)$ be the T-module with non-zero components $\mathsf{L}(-n)^0 = k^{n+1}$ and $\mathsf{L}(-n)^{-1} = k^n$ and action

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, $\mathsf{L}(0)$ is the residue field $k = \mathsf{T}/(\xi_1, \xi_2)$. It is easy to see that for $n > 0$ the T-module $\mathsf{L}(-n)[n]$ is a nth syzygy of k and is isomorphic to $\operatorname{Hom}_k\big(\mathsf{L}(n)[-n], k\big)$.

4.3.3. For each integer $n > 0$ and each $\lambda \in k \cup \{\infty\} = \mathbb{P}^1_k$, let $\mathsf{M}(n, \lambda)$ be the T-module with non-zero components $\mathsf{M}(n, \lambda)^1 = k^n$ and $\mathsf{M}(n, \lambda)^0 = k^n$ and action

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix} \quad \text{if } \lambda \neq \infty,$$

$$\xi_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{if } \lambda = \infty.$$

The significance of the modules just described is due to a classical result of Kronecker [25]: For $k = \mathbb{C}$, the pairs (ξ_1, ξ_2) appearing above form a complete list of indecomposable pairs of commuting complex matrices (ξ_1, ξ_2) up to similarity. Dieudonné [17] gave the first modern proof, over an arbitrary algebraically closed field. Heller and Reiner [22] reinterpreted the result as a description of the isomorphism classes of indecomposable (not necessarily graded) modules. We abstract:

4.4

The T-modules K, $\mathsf{L}(n)$, $\mathsf{M}(n, \lambda)$, with $n \in \mathbb{Z}$ and $\lambda \in \mathbb{P}^1_k$, are indecomposable. When k is algebraically closed, each finite graded T-module is isomorphic to a direct sum of shifts of these. Such a decomposition is unique in the sense of Krull–Schmidt.

4.5

By Subsection 3.12, the classification of indecomposable graded T-modules yields a classification of the indecomposable linear complexes over $\mathscr{R} = k[\chi_1, \chi_2]$.

4.5.1. The module $\mathsf{K} = \mathrm{Hom}_k(\mathsf{T}, k)$ corresponds to the Koszul complex

$$\mathscr{K} = \quad 0 \to \mathscr{R}\{-4\} \xrightarrow{\binom{-\chi_2}{\chi_1}} \mathscr{R}^2\{-2\} \xrightarrow{(\chi_1\ \chi_2)} \mathscr{R} \to 0.$$

4.5.2. For $n \geq 0$ the module $\mathsf{L}(n)$ of Subsection 4.3.3 corresponds to the complex

$$\mathscr{L}^\bullet(n) = \quad 0 \to \mathscr{R}^n\{-2\} \to \mathscr{R}^{n+1} \to 0$$

concentrated in complex degrees -1 and 0, with non-zero differential

$$\begin{pmatrix} \chi_1 & 0 & 0 & \cdots & 0 & 0 \\ \chi_2 & \chi_1 & 0 & \cdots & 0 & 0 \\ 0 & \chi_2 & \chi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \chi_1 & 0 \\ 0 & 0 & 0 & \cdots & \chi_2 & \chi_1 \\ 0 & 0 & 0 & \cdots & 0 & \chi_2 \end{pmatrix}.$$

Its only non-zero homology is in degree 0, where the Hilbert–Burch theorem yields $\mathscr{H}^0(\mathscr{L}^\bullet(n)) \cong \mathscr{L}(n)$, with $\mathscr{L}(n) = (\chi_1, \chi_2)^n\{2n\}$.

For $n > 0$ the module $\mathsf{L}(-n)$ of Subsection 4.3 corresponds to the complex

$$\mathscr{L}^\bullet(-n) = \quad 0 \to \mathscr{R}^{n+1} \to \mathscr{R}^n\{2\} \to 0$$

concentrated in complex degrees 0 and 1, with non-zero differential

$$\begin{pmatrix} \chi_1 & \chi_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \chi_1 & \chi_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \chi_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \chi_1 & \chi_2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \chi_1 & \chi_2 \end{pmatrix}.$$

As $\mathcal{L}^\bullet(-n) = \operatorname{Hom}_{\mathcal{R}}(\mathcal{L}^\bullet(n), \mathcal{R})$, we see that $\mathcal{H}^i(\mathcal{L}^\bullet(-n)) = 0$ for $i \neq 0, 1$, that $\mathcal{H}^0(\mathcal{L}^\bullet(-n)) \cong \mathcal{R}\{-2n\}$, and that $\mathcal{H}^1(\mathcal{L}^\bullet(-n)) \cong \mathcal{L}'(n)\{2\}$, where

$$\mathcal{L}'(n) = \operatorname{Hom}_k\left(\frac{\mathcal{R}}{(\chi_1, \chi_2)^n}, k\right)\{-2n+2\}.$$

4.5.3. For $n > 0$ and $\lambda \in \mathbb{P}^1_k$ the module $\mathsf{M}(n, \lambda)$ of Subsection 4.3.3 produces the complex

$$\mathcal{M}^\bullet(n, \lambda) = \quad 0 \to \mathcal{R}^n\{-2\} \to \mathcal{R}^n \to 0$$

concentrated in complex degrees -1 and 0 with non-zero differential

$$\begin{pmatrix} \chi_1+\lambda\chi_2 & 0 & 0 & \cdots & 0 & 0 \\ \chi_2 & \chi_1+\lambda\chi_2 & 0 & \cdots & 0 & 0 \\ 0 & \chi_2 & \chi_1+\lambda\chi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \chi_1+\lambda\chi_2 & 0 \\ 0 & 0 & 0 & \cdots & \chi_2 & \chi_1+\lambda\chi_2 \end{pmatrix} \qquad \text{if } \lambda \neq \infty,$$

$$\begin{pmatrix} \chi_2 & 0 & 0 & \cdots & 0 & 0 \\ \chi_1 & \chi_2 & 0 & \cdots & 0 & 0 \\ 0 & \chi_1 & \chi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \chi_2 & 0 \\ 0 & 0 & 0 & \cdots & \chi_1 & \chi_2 \end{pmatrix} \qquad \text{if } \lambda = \infty.$$

Clearly $\mathcal{H}^i(\mathcal{M}^\bullet(n, \lambda)) = 0$ for $i \neq 0$. We claim that $\mathcal{H}^0(\mathcal{M}^\bullet(n, \lambda))$ is isomorphic to

$$\mathcal{M}(n, \lambda) = \begin{cases} \dfrac{(\chi_1, \chi_2)^n}{(\chi_1+\lambda\chi_2)^n}\{2n\} & \text{if } \lambda \neq \infty, \\[2ex] \dfrac{(\chi_1, \chi_2)^n}{(\chi_2)^n}\{2n\} & \text{if } \lambda = \infty. \end{cases}$$

This amounts to the exactness of $\mathcal{M}^+(n, \lambda)$, the complex obtained by augmenting $\mathcal{M}^\bullet(n, \lambda)$ through the map $\epsilon_{n,\lambda}$ to $\mathcal{M}(n, \lambda)$ given on the canonical basis of $\mathcal{R}^n$ by

$$\epsilon_{n,\lambda}(e_i) = \begin{cases} (-\chi_2)^{n-i+1}(\chi_1+\lambda\chi_2)^{i-1} & \text{if } \lambda \neq \infty, \\ (-\chi_1)^{n-i+1}\chi_2^{i-1} & \text{if } \lambda = \infty. \end{cases}$$

We have $\mathscr{H}^1(\mathscr{M}^+(n,\lambda)) = 0$ because $\epsilon_{n,\lambda}$ is surjective. The Euler characteristic of $\mathscr{M}^+(n,\lambda)$ vanishes; hence so does that of its homology, proving $\mathscr{H}^0(\mathscr{M}^+(n,\lambda)) = 0$.

We return to the notation of Theorem 4.1.

The following observation uses a variation of the argument of Buchsbaum and Eisenbud for [13, (1.3)]; it is further generalized by Iyengar [23, (2.1)].

4.6

Let $\mathbf{E}$ be a minimal Q-free resolution of M with $E_n = 0$ for $n > 2$.

For $e = 1, 2$ choose a homotopy $\sigma^{(e)}\colon \mathbf{E} \to \mathbf{E}$ between $f_e \operatorname{id}^{\mathbf{E}}$ and $0^{\mathbf{E}}$. We have

$$\partial\sigma^{(j)}\sigma^{(i)} = f_j\sigma^{(i)} - \sigma^{(j)}\partial\sigma^{(i)} = f_j\sigma^{(i)} - \sigma^{(j)}f_i + \sigma^{(j)}\sigma^{(i)}\partial = f_j\sigma^{(i)} - f_i\sigma^{(j)}.$$

When $i = j$ the last expression vanishes. Otherwise, it equals $-\partial\sigma^{(i)}\sigma^{(j)}$ by symmetry. As ∂ is injective on E_2 we get $\sigma^{(j)}\sigma^{(i)}(a) = -\sigma^{(i)}\sigma^{(j)}(a)$ and $\sigma^{(e)2}(a) = 0$ for $a \in E_0$. The same relations hold trivially for $a \in E_n$ with $n \neq 0$, so $\xi_e(x) = \sigma^{(e)}(x)$ turns $\mathbf{E}$ into a DG module over $\mathbf{K} = R\langle\xi_1, \xi_2\rangle$ by Example 1.6.

4.7

In view of the minimality of the Koszul resolution $\mathbf{E}$ constructed above and of Subsection 3.7, the complex of graded $\mathscr{R}$-modules

$$\mathscr{C}^\bullet(M,k) = 0 \to \mathscr{R} \otimes_k \mathsf{M}^2\{-4\} \to \mathscr{R} \otimes_k \mathsf{M}^1\{-2\} \xrightarrow{\partial} \mathscr{R} \otimes_k \mathsf{M}^0 \to 0$$

concentrated in degrees -2, -1, and 0, computes $\operatorname{Ext}_R^*(M,k)$ by the formulas

$$\operatorname{Ext}_R^{\text{even}}(M,k) = \mathscr{H}^0(\mathscr{C}^\bullet(M,k)) \oplus \mathscr{H}^{-2}(\mathscr{C}^\bullet(M,k))\{2\},$$

$$\operatorname{Ext}_R^{\text{odd}}(M,k) = \mathscr{H}^{-1}(\mathscr{C}^\bullet(M,k))\{1\}.$$

When k is algebraically closed, Subsections 1.1 and 4.4 uniquely define

$$\begin{aligned} &\text{integers:} && p,\ q(e),\ r(e),\ s(e),\ t(0),\ t(1),\ t(2) \geq 0,\\ &\text{integers:} && a_1^e,\ \cdots,\ a_{q(e)}^e,\ b_1^e,\ \cdots,\ b_{r(e)}^e,\ c_1^e,\ \cdots,\ c_{s(e)}^e > 0,\\ &\text{pairs:} && (c_1^e, \lambda_1^e),\ \cdots,\ (c_{s(e)}^e, \lambda_{s(e)}^e) \quad \text{with} \quad \lambda_j^e \in \mathbf{P}_k^1, \end{aligned}$$

for $e = 0, 1$, such that the graded T-module $\mathsf{M} = \operatorname{Ext}_Q^*(M,k)$ is isomorphic to

$$\mathsf{K}^p \oplus \bigoplus_{h=1}^{q(0)} \mathsf{L}(-a_h^0)[-1] \oplus \bigoplus_{i=1}^{r(0)} \mathsf{L}(b_i^0) \oplus \bigoplus_{j=1}^{s(0)} \mathsf{M}(c_j^0, \lambda_j^0) \oplus k^{t(0)} \oplus k^{t(2)}[-2]$$
$$\oplus \left(\bigoplus_{h=1}^{q(1)} \mathsf{L}(-a_h^1)[-1] \oplus \bigoplus_{i=1}^{r(1)} \mathsf{L}(b_i^1) \oplus \bigoplus_{j=1}^{s(1)} \mathsf{M}(c_j^1, \lambda_j^1) \oplus k^{t(1)}\right)[-1].$$

The decomposition of M splits $\mathscr{C}^\bullet(M, k)$ into a direct sum of complexes of graded $\mathscr{R}$-modules isomorphic to shifts of the complexes $\mathscr{K}^\bullet$, $\mathscr{L}^\bullet(n)$, and $\mathscr{M}^\bullet(n, \lambda)$ of Subsection 4.5. Thus, the graded $\mathscr{R}$-module $\operatorname{Ext}_R^{\text{even}}(M, k)$ is isomorphic to

$$k^p \oplus \bigoplus_{h=1}^{q(0)} \mathscr{L}'(a_h^0) \oplus \bigoplus_{i=1}^{r(0)} \mathscr{L}(b_i^0) \oplus \bigoplus_{j=1}^{s(0)} \mathscr{M}(c_j^0,\ \lambda_j^0) \oplus \mathscr{R}^{t(0)} \oplus \mathscr{R}^{t(2)}\{-2\}$$
$$\oplus \bigoplus_{h=1}^{q(1)} \mathscr{R}\{-2a_h^1 - 2\}$$

and the graded $\mathscr{R}$-module $\operatorname{Ext}_R^{\text{odd}}(M, k)$ is isomorphic to

$$\left(\bigoplus_{h=1}^{q(1)} \mathscr{L}'(a_h^1) \oplus \bigoplus_{i=1}^{r(1)} \mathscr{L}(b_i^1) \oplus \bigoplus_{j=1}^{s(1)} \mathscr{M}(c_j^1,\ \lambda_j^1) \oplus \mathscr{R}^{t(1)}\right)\{-1\}$$
$$\oplus \bigoplus_{h=1}^{q(0)} \mathscr{R}\{-2a_h^0 - 1\}\ .$$

The multiplicities in the decompositions above satisfy some subtle relations. The first one comes from a well-known result of Auslander and Buchsbaum [3, (6.2)]:

4.8

As M is annihilated by a Q-regular element, we have $\sum_{n=0}^{q}(-1)^n \beta_n^Q(M) = 0$.

4.9

The numerical invariants appearing in Subsection 4.7 satisfy

$$q(0) + r(1) + t(1) = q(1) + r(0) + t(0) + t(2).$$

If ℓ denotes either side of the equality, then one of the following holds,

$$\ell = 0 = s(0) = s(1),$$
$$\ell = 0 < c(0) = c(1),$$
$$\ell > a(1) - a(0) + b(1) - b(0) + c(1) - c(0) + q(0) + t(1) - t(0) > 0,$$

with $a(e) = \sum_{h=1}^{q(e)} a_h^e$, $b(e) = \sum_{j=1}^{r(e)} b_j^e$, $c(e) = \sum_{j=1}^{s(e)} c_j^e$ for $e = 0, 1$.

It is important to note that $\ell = 1$ is excluded as a possibility, and it is certainly interesting to know whether there are further restrictions.

To establish these relations, note that the decomposition of $\mathrm{Ext}_Q^*(M,k)$ yields

$$\begin{aligned}\beta_0^Q(M) &= p + a(0) + b(0) + r(0) + c(0) + t(0),\\ \beta_1^Q(M) &= 2p + a(0) + q(0) + b(0) + c(0) + a(1) + b(1) + r(1)\\ &\qquad + c(1) + t(1),\\ \beta_2^Q(M) &= p + a(1) + q(1) + b(1) + c(1) + t(2).\end{aligned}$$

The expressions for ℓ follow from the equality in Subsection 4.8.

On the other hand, the decomposition of $\mathrm{Ext}_R^*(M,k)$ yields for $u \gg 0$ equalities

$$\begin{aligned}\beta_{2u}^R(M) &= \ell u - a(1) + b(0) + c(0) + t(0),\\ \beta_{2u+1}^R(M) &= \ell u - a(0) + b(1) + c(1) + q(0) + t(1).\end{aligned}$$

When $\ell = 0$ the Betti sequence of M is bounded, so by [5, (4.1)] it is eventually constant. If that constant is zero, we get the first relation; otherwise for $u \gg 0$ the formulas above give $c(0) = \beta_{2u}(M)$ and $c(1) = \beta_{2u+1}^R(M)$, and hence $c(0) = c(1) > 0$. When $\ell > 0$ the Betti sequence of M is unbounded, by [9, (7.8)] it is eventually strictly increasing; the formulas above translate the inequalities $\beta_{2u+2}^R(M) > \beta_{2u+1}^R(M) > \beta_{2u}^R(M)$ into the desired inequalities for ℓ.

4.10

For M as in Subsection 4.7, Proposition 1.2 yields

$$\text{f-rank}_R M = \text{f-rank}_{\mathsf{T}}\, \mathrm{Ext}_Q^*(M,k) = p.$$

The free rank of M can also be read off the action of $\mathscr{R}$ on $\mathrm{Ext}_R^{\leqslant 3}(M,k)$,

$$\begin{aligned}&\text{f-rank}_R M\\ &\quad = \operatorname{rank}_k \operatorname{Ker}\left(\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}\colon \mathrm{Ext}_R^0(M,k) \to \mathrm{Ext}_R^2(M,k) \oplus \mathrm{Ext}_R^2(M,k)\right)\\ &\qquad - \operatorname{rank}_k \operatorname{Coker}\Big((\chi_1\ \chi_2)\colon \mathrm{Ext}_R^1(M,k) \oplus \mathrm{Ext}_R^1(M,k)\\ &\qquad\qquad \to \mathrm{Ext}_R^3(M,k)\Big).\end{aligned}$$

Indeed, if a' is the number of direct summands of $\mathrm{Ext}_R^*(M,k)$ isomorphic to $\mathscr{R}\{-3\}$, then by Subsection 4.7 the kernel has rank $p + a'$ and the cokernel has rank a'.

Proof of Theorem 4.1. Recall that M is a finite R-module with $\operatorname{proj\,dim}_Q M$ finite and $g = \operatorname{depth} R - \operatorname{depth}_R M$. We want to prove that the $\mathcal{R}$-module $\operatorname{Ext}_R^*(M, k)$ is generated in degrees $\leq m$, where $m = \max\{2\beta_g^R(M) + 1\, , \, 2\beta_g^R(M) + 2\} + g$.

A gth syzygy of M over R has finite projective dimension over Q and depth equal to that of R; by Subsection 3.6 its cohomology over R is isomorphic to a shift of $\operatorname{Ext}_R^{\geqslant g}(M, k)$. Switching from M to its syzygy, we may assume that $g = 0$.

Choose a faithfully flat local homomorphism $Q \to Q'$ such that $\mathfrak{n}Q'$ is the maximal ideal of Q' and $k' = Q'/\mathfrak{n}Q'$ is algebraically closed, and set $R' = R \otimes_Q Q' = Q'/(\boldsymbol{f})$, where $\boldsymbol{f}'$ is the image of $\boldsymbol{f}$ in Q'. For $M' = M \otimes_Q Q' = M \otimes_R R'$ we then have a natural isomorphism of graded k'-vector spaces

$$\operatorname{Ext}_R^*(M, k) \otimes_k k' \cong \operatorname{Ext}_{R'}^*(M', k').$$

If $\mathcal{R}'$ is the ring of cohomology operators defined by the Q'-regular sequence $\boldsymbol{f}'$, then the construction of the operators in Section 3 shows that this map is equivariant over an isomorphism $\mathcal{R} \otimes_k k' \cong \mathcal{R}'$. Switching from Q to Q' and changing notation once more, we may assume that the residue field k is algebraically closed. The desired assertion is then obviously contained in the formulas in Subsection 4.7. ∎

5. MINIMAL RESOLUTIONS

In this section $(Q, \mathfrak{m}, k)$ is a local ring, $\boldsymbol{f} = f_1, \ldots, f_c$ is a Q-regular sequence, $R = Q/(\boldsymbol{f})$, and M is a finite R-module of finite projective dimension over Q. We explore the discrepancy between the R-free resolutions $\mathrm{G}(\mathbf{E})$ constructed in Theorem 2.4 from Koszul resolutions $\mathbf{E}$ of M over Q and its *minimal* R-free resolution $\mathbf{F}$.

It is clear that $\mathrm{G}(\mathbf{E})$ itself is minimal if and only if $\mathbf{E}$ is a minimal free resolution of M over Q and each ξ_j acts trivially on $\operatorname{Ext}_Q^*(M, k)$ (equivalently, on $\operatorname{Tor}_*^Q(M, k)$). However, the first condition is known to fail in general; (cf. [4] or [7, Sect. 2]). When it does hold the second one usually fails. On the positive side, we note that M always has a Koszul resolution of length equal to $\operatorname{proj\,dim}_Q M$; cf. Subsection 2.1.

5.1

Lifting id^M to quasi-isomorphisms α: $\mathbf{F} \to \mathrm{G}(\mathbf{E})$ and β: $\mathrm{G}(\mathbf{E}) \to \mathbf{F}$ one gets R-linear chain maps $\beta\chi_j\alpha$: $\mathbf{F} \to \mathbf{F}$ of degree -2. Identifying $\mathrm{H}^* \operatorname{Hom}_R(\mathbf{F}, k)$ with $\operatorname{Ext}_R^*(M, k)$ we see that $\mathrm{H}\operatorname{Hom}_R(\beta\chi_j\alpha, k)$ coincides with the action of χ_j.

The homotopy class of each map $\beta\chi_j\alpha$ is independent of the choices of α, β, or $\mathbf{E}$; cf. Subsection 3.2. If there exist chain maps χ'_j that are homotopic to $\beta\chi_j\alpha$ and satisfy $\chi'_i\chi'_j = \chi'_j\chi'_i$ for $1 \le i, j \le c$, then we say that $\mathbf{F}$ has a *proper* structure of DG module over $\mathcal{S}$. When $c = 1$ this condition is vacuous; in general, when it holds one gets on $\mathbf{F}$ a structure of DG module over $\mathcal{S}$ by letting χ_j act as χ'_j.

5.2

A stronger requirement is that α can be chosen to satisfy $\chi_j\alpha(\mathbf{F}) \subseteq \alpha(\mathbf{F})$ for $1 \le j \le c$; we then say that $\mathbf{F}$ *embeds as a DG submodule* of $\mathrm{G}(\mathbf{E})$; since $\mathbf{F}$ is minimal α defines an isomorphism $\mathbf{F} \cong \alpha(\mathbf{F})$ that induces a proper DG module structure on $\mathbf{F}$. We make some general remarks on the existence of such embeddings.

5.2.1. Let $M = R$ and let $\mathbf{K}$ be a Koszul complex resolving R over Q. The inclusion $R \otimes 1 \subseteq \mathrm{G}(\mathbf{K})$ is then a quasi-isomorphism by Theorem 2.4. It is obvious that this map splits over R and that its image is a DG submodule over $\mathcal{S}$.

5.2.2. Let $M = M' \oplus M''$. If $\mathbf{E}^{(i)}$ is a Koszul resolution of $M^{(i)}$ and $\mathbf{F}^{(i)}$ is a minimal R-free resolution of $M^{(i)}$ that embeds as a DG submodule of $\mathrm{G}(\mathbf{E}^{(i)})$, then the minimal R-free resolution $\mathbf{F}' \oplus \mathbf{F}''$ of M embeds as a DG submodule over $\mathcal{S}$ of $\mathrm{G}(\mathbf{E}') \oplus \mathrm{G}(\mathbf{E}'') \cong \mathrm{G}(\mathbf{E}' \oplus \mathbf{E}'')$.

5.2.3. The minimal resolutions of the modules M_s described in Subsection 4.2 admit no embedding as DG submodules of $\mathrm{G}(\mathbf{E})$: this follows from the result of [9, (9.1)] that if such an an embedding exists, then the graded $\mathcal{R}$-module $\mathrm{Ext}^*_R(M, k)$ is generated in degrees $\le \operatorname{proj\,dim}_Q M$.

The next theorem is a partial converse to the result of [9] mentioned above.

5.3. THEOREM. *Let M be a finite module over $R = Q/(f_1, f_2)$.*
If $\operatorname{proj\,dim}_Q M \le 2$ *then M has a minimal Koszul resolution* $\mathbf{E}$ *over Q.*
If furthermore $\mathrm{Ext}^*_R(M, k)$ *is generated over $\mathcal{R}$ in degree ≤ 2, then* $\mathbf{F}$ *is isomorphic to a DG submodule of* $\mathrm{G}(\mathbf{E})$ *over $\mathcal{S}$.*

Eisenbud [18, p. 37] conjectured that the minimal resolution of each finite R-module $\mathbf{F}$ can be embedded as a DG submodule of an R-free resolution of M constructed from a system of higher homotopies on some Q-free resolution of M; cf. Subsection 2.5. In [9, (9.3)] the conjecture is shown to fail, for the same reason as given in Subsection 5.2.3, but it is still not known whether the conjecture can also fail for all high syzygies of a finite R-module.

In codimension 1 the asymptotic conjecture holds for nth syzygies with $n >$ depth R by Subsection 2.7. We prove it in codimension 2 with an *effective* bound on n that depends on the module M. The modules M_s show that no *universal* bound exists. In codimension ≥ 3 it is not known even whether $\mathbf{F}_{>n}$ for $n \gg 0$ admits a DG module structure over $\mathcal{S}$ that is proper in the sense of Subsection 5.1.

5.4. COROLLARY. *Let M be a finite module over $R = Q/(f_1, f_2)$ with* depth $R -$ depth$_R$ $M = g$. *If* proj dim$_Q$ $M < \infty$ *and M' is an nth syzygy of M with*

$$n > \max\left\{2\beta_g^R(M) - 1\,,\; 2\beta_{g+1}^R(M)\right\} + g - 1\,,$$

then $\mathbf{F}_{>n}$ is a DG submodule of $\mathrm{G}(\mathbf{E}')$ *for a finite Koszul resolution* $\mathbf{E}'$ *of M'.*

Proof. By Subsection 3.6, $\mathrm{Ext}_R^{\geqslant n}(M, k)[n]$ is isomorphic to the cohomology of the nth syzygy of M, so combine the preceding theorem with Subsection 4.7. ∎

Proof of Theorem 5.3. We note that the Koszul resolution $\mathbf{E}$ of M from Subsection 4.6 is minimal, set $\mathbf{E}^\dagger = \mathrm{Hom}_Q(\mathbf{E}, R)$, and use overlines for reduction modulo $\mathfrak{n}$.

As the $\mathcal{R}$-module $\mathrm{Ext}_R^*(M, k)$ is generated in degrees ≤ 2, we see from Subsection 4.7 that it has no direct summand isomorphic to $\mathcal{L}'(a)$; this means that $\mathrm{Ext}_Q^*(M, k) = \overline{\mathbf{E}}^\dagger$ has no direct summand isomorphic to $\mathsf{L}(-a)$ with $a > 0$. By Subsection 5.2.2 and Subsection 5.2.1 we may further assume that M has no direct summand isomorphic to R, and from Proposition 1.2 we see that $\overline{\mathbf{E}}^\dagger$ then has no direct summand of the form K.

Fix bases of the vector spaces $\mathsf{L}(b_i^1)[-1]^2$ for $1 \leq i \leq r(1)$, and $\mathsf{M}(c_j^1, \lambda_j^1)[-1]^2$ for $1 \leq j \leq s(1)$ and let Y_2 be a lifting of that basis to $E_2^\dagger$. Similarly, choose bases in $\mathsf{L}(b_i^0)^1$ for $1 \leq i \leq r(0)$ and in $\mathsf{M}(c_j^0, \lambda_j^0)^1$ for $1 \leq j \leq s(0)$ and lift them to a subset $\mathrm{Y}_1 \subset E_1^\dagger$. It follows from Subsection 4.5 that the subcomplex

$$\mathcal{D}^\bullet = \quad 0 \to \mathcal{R}\,\overline{\mathrm{Y}}_2 \to \mathcal{R}\,\partial(\overline{\mathrm{Y}}_2) \oplus \mathcal{R}\,\overline{\mathrm{Y}}_1 \to \mathcal{R}\,\partial(\overline{\mathrm{Y}}_1) \to 0$$

of graded $\mathcal{R}$-modules of $\mathcal{C}^\bullet = \mathrm{C}(\mathbf{E}, k) \cong \mathbf{C} \otimes_R k$ is exact, where $\mathbf{C} = \mathrm{C}(\mathbf{E}R,)$.

Recall that $\mathbf{C}^\natural = \oplus_{p=0}^2 \mathcal{S} \otimes_R E_p^\dagger$ and let $\mathbf{D}$ be the DG submodule of $\mathbf{C}$ generated over $\mathcal{S}$ by $1 \otimes \mathrm{Y}_e$ and $1 \otimes \partial(\mathrm{Y}_e)$ for $e = 1, 2$. Let $\mathbf{D}^\natural$ be the graded R-module generated by $\chi^{n_1}\chi^{n_2} \otimes v$ and $\partial(\chi^{n_1}\chi^{n_2} \otimes v)$ with $n_1 \geq 0$, $n_2 \geq 0$, $v \in \mathrm{Y}_1 \cup \mathrm{Y}_2$. These elements are linearly independent modulo $\mathfrak{m}\mathbf{C}$, so they form a basis of $\mathbf{D}^\natural$ over R and $\mathbf{D}$ is a direct summand of $\mathbf{C}$ as a complex of R-modules. The exactness of $\mathcal{D}^\bullet$ implies that $\mathbf{D}$ is split exact.

Set $\mathbf{F} = \mathrm{Hom}_R(\mathbf{C}/\mathbf{D}, R)$ and note that $\mathrm{Hom}_R(\mathbf{C}, R)$ is naturally isomorphic as DG module over $\mathcal{S}$ to the R-free resolution $\mathrm{G}(\mathbf{E})$ of M from Theorem 2.4. Dualizing the exact sequence $0 \to \mathbf{D} \to \mathbf{C} \to \mathbf{C}/\mathbf{D} \to 0$ we get an exact sequence

$$0 \to \mathbf{F} \to \mathrm{G}(\mathbf{E}) \to \mathrm{Hom}_R(\mathbf{D}, R) \to 0$$

of DG modules over $\mathcal{S}$ in which $\mathbf{F}$ splits off as a complex of R-modules.

The complex $\mathrm{Hom}_R(\mathbf{D}, R)$ is split exact along with $\mathbf{D}$, so the sequence shows that $\mathrm{H}_*(\mathbf{F}) \cong \mathrm{H}_*(\mathrm{G}(\mathbf{E})) \cong M$. The differential of the complex $\mathcal{C}/\mathcal{D}$ is trivial by the choice of $\mathcal{D}$. As $\mathcal{C}/\mathcal{D} \cong (\mathbf{C}/\mathbf{D}) \otimes_R k$, we see that $\partial(\mathbf{F}) \subseteq \mathfrak{m}\mathbf{F}$. Thus, $\mathbf{F}$ is a minimal free resolution of M and a DG submodule of $\mathrm{G}(\mathbf{E})$ over $\mathcal{S}$. ∎

Assume that one is able to construct *finite* sequences of syzygies over a residue ring R of codimension 2 over Q. When R is *not* a complete intersection, Iyengar [23, (3.4)] constructs a *minimal* R-free resolution of an R-module M, starting with a minimal Q-free resolution of its (depth $R -$ depth$_R$ $M + 2$)nd syzygy. The preceding arguments yield a similar recipe that works when R is a complete intersection.

5.5

Let L be a finite module over $R = Q/(f_1, f_2)$ with $\operatorname{proj\,dim}_Q L < \infty$.

Set $g = \operatorname{depth} R - \operatorname{depth}_R L$ and construct the beginning of a minimal resolution of L up to the gth syzygy,

$$0 \to L' \to F_{g-1} \xrightarrow{\partial_{n-2}} F_{g-2} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to L \to 0.$$

The R-module L' is free precisely when $\operatorname{proj\,dim}_R L$ is finite. If that is not the case, set $n = \max\left\{2\beta_0^R(L'),\ 2\beta_1^R(L') + 1\right\}$ and extend the minimal resolution to

$$0 \to M \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_{g+1} \xrightarrow{\partial_{g+1}} F_g \to L' \to 0.$$

Note that $\operatorname{proj\,dim}_Q M = 2$, form a minimal Q-free resolution

$$0 \to E_2 \to E_1 \xrightarrow{\alpha} E_0 \to M \to 0,$$

and let u, v, w, be the ranks of E_0, E_1, E_2. Choosing homotopies from $f_1 \,\mathrm{id}^{\mathbf{E}}$, $f_2 \,\mathrm{id}^{\mathbf{E}}$ to 0 we get R-linear maps $\beta^{(1)}, \beta^{(2)}\colon E_0 \to E_1$ and $\gamma^{(1)}, \gamma^{(2)}\colon E_1 \to E_2$.

Let A, $B^{(1)}$, $B^{(2)}$, $C^{(1)}$, $C^{(2)}$, be, respectively, the matrices of the homomorphisms $\alpha \otimes_Q R$, $\beta^{(1)} \otimes_Q R$, $\beta^{(2)} \otimes_Q R$, $\gamma^{(1)} \otimes_Q R$, $\gamma^{(2)} \otimes_Q R$ in some bases of the free R-modules $E_q \otimes_Q R$, subject to the only restriction that the last $v - v'$ rows of $B = \left(B^{(1)} \,\middle|\, B^{(2)}\right)$ form a k-basis of the row-space of $\overline{B}$. We describe the infinite tail $\mathbf{F}_{>0}$ of the minimal resolution of M in terms of the five matrices above.

Let $\mathbf{C}'$ be a free graded R-module with basis

$$\left\{ \begin{array}{l|cc} \chi_1^{n_1}\chi_2^{n_2}v_i^0 & & \\ \qquad \text{with } \left|\chi_1^{n_1}\chi_2^{n_2}v_i^0\right| = 2n_1+2n_2 & 1 \le i \le v' & \\ \chi_1^{n_1}\chi_2^{n_2}v_h^1 & 1 \le h \le u & \text{and } n_1, n_2 \ge 0 \\ \qquad \text{with } \left|\chi_1^{n_1}\chi_2^{n_2}v_h^1\right| = 2n_1+2n_2+1 & & \end{array} \right\}.$$

The following set generates a homogeneous R-module direct summand $\mathbf{D}'$ of $\mathbf{C}'$,

$$\left\{ \begin{array}{l|cc} \sum_{h=1}^{u} b_{ih}^{(1)}\chi_1^{n_1+1}\chi_2^{n_2}v_h^1 & & \\ \qquad + \sum_{h=1}^{u} b_{ih}^{(2)}\chi_1^{n_1}\chi_2^{n_2+1}v_h^1 & v' < i \le v & \\ \sum_{i=1}^{v'} c_{ji}^{(1)}\chi_1^{n_1+1}\chi_2^{n_2}v_i^0 & 1 \le j \le w & \text{and } n_1, n_2 \ge 0 \\ \qquad + \sum_{i=1}^{v'} c_{ji}^{(2)}\chi_1^{n_1}\chi_2^{n_2+1}v_i^0 & & \end{array} \right\}.$$

The formulas below define on $\mathbf{F}' = \mathbf{C}'/\mathbf{D}'$ an R-linear differential of degree one,

$$\begin{aligned} \partial(\chi_1^{n_1}\chi_2^{n_2}v_h^0) &= \sum_{i=1}^{v'} a_{hi}\chi_1^{n_1}\chi_2^{n_2}v_i^1 \\ \partial(\chi_1^{n_1}\chi_2^{n_2}v_i^1) &= \sum_{h=1}^{u} b_{ji}^{(1)}\chi_1^{n_1+1}\chi_2^{n_2}v_h^0 + \sum_{h=1}^{u} b_{ji}^{(2)}\chi_1^{n_1}\chi_2^{n_2+1}v_h^0. \end{aligned}$$

The R-dual of the complex $(\mathbf{F}', \partial)$ is a minimal free resolution of M.

6. SPECTRAL SEQUENCE

Let $\boldsymbol{f} = \{f_1, \ldots, f_c\}$ be a Koszul-regular set in a commutative ring Q, let $R = Q/(\boldsymbol{f})$, and let $\mathcal{S} = R[\chi_1, \ldots, \chi_c]$ be the ring of cohomology operators defined by $\boldsymbol{f}$. This section describes our main tool for computing cohomology operators—a spectral sequence of $\mathcal{S}$-modules that approximates $\operatorname{Ext}_R^*(\ ,\ )$. The result is only stated in cohomology, as its homological version is not used here.

6.1. THEOREM. *For all R-modules M, N there exists a spectral sequence*

$${}^2\mathrm{E}^{p,q} = \mathcal{S}^{2p+q} \otimes_R \operatorname{Ext}_Q^{-p}(M, N) \Longrightarrow \operatorname{Ext}_R^{p+q}(M, N)$$

with differentials ${}^r\mathrm{d}$ *of degree* 1 *for the cohomological degree* $p+q$, *and*

$${}^r\mathrm{d}^{p,q}\colon {}^r\mathrm{E}^{p,q} \to {}^r\mathrm{E}^{p+r-1,\,q-r+2}.$$

The spectral sequence is natural in both modules M *and* N.
The rows ${}^r\mathrm{E}^{*,q}$ *are equal to* 0 *when* q *is odd, so*

$${}^r\mathrm{d} = 0 \qquad \textit{and} \qquad {}^r\mathrm{E} = {}^{r+1}\mathrm{E} \qquad \textit{when } r \textit{ is odd}.$$

The columns ${}^r\mathrm{E}^{p,*}$ *form complexes of graded* $\mathcal{S}$*-modules*

$$\chi_j\colon {}^r\mathrm{E}^{p,q} \to {}^r\mathrm{E}^{p,q+2} \quad \textit{and} \quad \chi_j\,{}^r\mathrm{d} = {}^r\mathrm{d}\,\chi_j.$$

The page ${}^2\mathrm{E}$ *is the complex* $\mathscr{C}^\bullet(M,N)$ *of graded* $\mathcal{S}$*-modules*

$$\cdots \to \mathcal{S}\otimes_R \operatorname{Ext}_Q^{-p}(M,N)\{2p\} \xrightarrow{\mathrm{d}} \mathcal{S}\otimes_R \operatorname{Ext}_Q^{-p+1}(M,N)\{2p-2\} \to \cdots$$

with differential d *given by multiplication with* $\sum_{j=1}^{c} \chi_j \otimes_k \xi_j$, *and action of* Λ

$$\xi_j\colon {}^2\mathrm{E}^{p,q} \to {}^2\mathrm{E}^{p+1,\,q-2} \qquad \textit{such that} \quad \xi_j\mathrm{d} = -\mathrm{d}\xi_j \qquad \textit{and} \quad \xi_j\chi_i = -\chi_i\xi_j.$$

The abutment $\operatorname{Ext}_R^*(M,N)$ *has a filtration by graded* $\mathcal{S}$*-submodules and*

$$\bigoplus_{p,q} {}^\infty\mathrm{E}^{p,q} \cong {}^0\mathrm{E}(\operatorname{Ext}_R^*(M,N))$$

is an isomorphism of bigraded $\mathcal{S}$*-modules.*

It is clear from the description of the second page of the spectral sequence that outside of the sector $p \le 0$ and $2p+q \ge 0$ the sequence satisfies ${}^r\mathrm{E}^{p,q} = 0$ for all $r \ge 2$. The resulting useful edge homomorphisms are described in Subsection 6.3.

As an illustration we give a transparent proof of the main result of [21] (the converse holds as well and is proved in [9]). Note that $\operatorname{Ext}_Q^*(M,N)$ (respectively, $\operatorname{Ext}_R^*(M,N)$) is killed by the annihilators of M and N in Q (respectively, R) so both Ext's are naturally graded modules over the ring

$$\overline{R} = Q/(\operatorname{ann}_Q M + \operatorname{ann}_Q N) = R/(\operatorname{ann}_R M + \operatorname{ann}_R N).$$

The next corollary was initially proved by Gulliksen [21] (he uses a different construction of the operators, but they coincide with those considered here, due to [11]).

6.2. Corollary. *If the* $\overline{R}$*-module* $\operatorname{Ext}_Q^*(M,N)$ *is noetherian, then* $\operatorname{Ext}_R^*(M,N)$ *is a noetherian graded module over* $\overline{R}[\chi_1,\ldots,\chi_c] = \overline{R}\otimes_R \mathcal{S}$.

Proof. The hypotheses imply that ${}^2\mathrm{E}$ is a bigraded noetherian $\mathcal{S}$-module, and ${}^2\mathrm{E}^{p,*} = 0$ for $|p| \gg 0$. It follows that ${}^r\mathrm{E}$ is a bigraded noetherian $\mathcal{S}$-module for each $r \geq 2$, and ${}^r\mathrm{E} = {}^\infty\mathrm{E}$ for $r \gg 0$. Thus, $\operatorname{Ext}^*_R(M, N)$ is filtered by graded $\mathcal{S}$-submodules and the associated bigraded module is noetherian. We conclude that $\operatorname{Ext}^*_R(M, N)$ is a noetherian graded module over $\mathcal{S}$, and hence over $\overline{R} \otimes_R \mathcal{S}$. ∎

Proof of Theorem 6.1. Let $\mathbf{E}$ be a Koszul resolution of M. For the DG module $\mathrm{C}(\mathbf{E}, N)$ with $\mathrm{C}(\mathbf{E}, N)^\natural = \oplus_{u=0}^\infty \mathcal{S} \otimes_R \operatorname{Hom}_Q(E_u, N)\{-2u\}$ from Subsection 3.2 set

$$F(p) = \bigoplus_{\substack{u+v \geqslant -p \\ u \geqslant 0}} \mathcal{S}^v \otimes_R \operatorname{Hom}_Q(E_u, N)\{-2u\}.$$

This is a descending filtration of $\mathrm{C}(\mathbf{E}, N)$, graded by cohomological degree, by means of DG submodules over $\mathcal{S}$. It defines a spectral sequence with differentials ${}^r\mathrm{d}^{p,q}\colon {}^r\mathrm{E}^{p,q} \to {}^r\mathrm{E}^{p+r-1,\, q-r+2}$. Since $\mathrm{C}(\mathbf{E}, N) \cong \operatorname{Hom}_R(\mathrm{G}(\mathbf{E}), N)$, where $\mathrm{G}(\mathbf{E})$ is the R-free resolution of M given by Theorem 2.4, the spectral sequence converges to $\operatorname{Ext}^*_R(M, N)$ for the cohomological degree $p + q$. Its 0th page has the form

$${}^0\mathrm{E}^{p,q} = \big(F(p)/F(p-1)\big)^{p+q} = \mathcal{S}^{2p+q} \otimes_P \operatorname{Hom}_Q(E_{-p}, N)$$

with differential ${}^0\mathrm{d}^{p,q} = \mathcal{S}^{2p+q} \otimes_R \operatorname{Hom}_Q(\partial^{\mathbf{E}}_{-p}, N)$. As $\mathcal{S}^{2p+q}$ is a free R-module, we have ${}^2\mathrm{E}^{p,q} = {}^1\mathrm{E}^{p,q} = \mathcal{S}^{2p+q} \otimes_R \operatorname{Ext}_Q^{-p}(M, N)$ and ${}^2\mathrm{d}^{p,q} = \sum_{j=1}^c \chi_j \otimes \xi_j$.

When q is odd ${}^0\mathrm{E}^{p,q} = 0$. It follows that if r is odd then ${}^r\mathrm{d} = 0$, so ${}^{r+1}\mathrm{E} = {}^r\mathrm{E}$. Furthermore, the construction shows that the algebra $\mathcal{S} \otimes_R \Lambda$ operates on the spectral sequence by $\chi_j \cdot {}^r\mathrm{E}^{p,q} \subseteq {}^r\mathrm{E}^{p,q+2}$ and $\xi_j \cdot {}^r\mathrm{E}^{p,q} \subseteq {}^r\mathrm{E}^{p+1,q-2}$. ∎

6.3

The preceding proof yields useful information on the edge homomorphisms of the spectral sequence of Theorem 6.1, determined by its vanishing lines

$${}^r\mathrm{E}^{p,q} = 0 \qquad \text{when} \quad 2p + q < 0 \qquad \text{or} \qquad p > 0.$$

6.3.1. The first vanishing line defines diagonal edge homomorphisms

$$\operatorname{Ext}^u_R(M, N) \twoheadrightarrow {}^\infty\mathrm{E}^{-u,2u} \hookrightarrow \cdots \hookrightarrow {}^3\mathrm{E}^{-u,2u} \hookrightarrow {}^2\mathrm{E}^{-u,2u} = \operatorname{Ext}^u_Q(M, N);$$

their composition is the component ρ^u_{MN} of the change-of-rings morphism from Subsection 3.5.1 and ${}^4\mathrm{E}^{-u,2u} = {}^3\mathrm{E}^{-u,2u} = \operatorname{ann}_{\operatorname{Ext}^u_Q(M,N)}(\xi_1, \ldots, \xi_c)$.

6.3.2. The second vanishing line defines vertical edge homomorphisms

$$\mathscr{S}^{2v} \otimes_R \operatorname{Ext}_Q^0(M, N) = {}^2\mathrm{E}^{0,\,2v} \twoheadrightarrow {}^3\mathrm{E}^{0,\,2v} \twoheadrightarrow \cdots \twoheadrightarrow {}^\infty\mathrm{E}^{0,\,2v} \hookrightarrow \operatorname{Ext}_R^{2v}(M, N);$$

their composition is the component $\varkappa_{MN}^{2v}$ of the morphism from Subsection 3.5.2.

6.3.3. If $\operatorname{Ext}_Q^n(M, N) = 0$ for $n > g$, then ${}^r\mathrm{E}^{p,\,q} = 0$ for $p < -g$; it follows that ${}^r\mathrm{E}^{p,\,q} = {}^\infty\mathrm{E}^{p,\,q}$ for $r > g + 1$ and so there exist vertical edge homomorphisms

$$\begin{aligned}\operatorname{Ext}_R^v(M, N) \twoheadrightarrow {}^\infty\mathrm{E}^{-g,\,v+g} \hookrightarrow \cdots \hookrightarrow {}^3\mathrm{E}^{-g,\,v+g} \hookrightarrow {}^2\mathrm{E}^{-g,\,v+g} \\ = \mathscr{S}^v \otimes_R \operatorname{Ext}_Q^g(M, N);\end{aligned}$$

their composition is a component of a morphism of graded $\mathscr{S}$-modules

$$\operatorname{Ext}_R^{\geqslant g}(M, N) \to \mathscr{S} \otimes_R \operatorname{Ext}_Q^g(M, N)\{-g\}.$$

6.4

The spectral sequence allows for some extensions of earlier results.

6.4.1. If $\boldsymbol{f} = f_1$ and $\operatorname{Ext}_Q^p(M, N) = 0$ for $p > 1$, then the line ${}^2\mathrm{E}^{*,\,2v}$ of the spectral sequence of Theorem 6.1 becomes the complex

$$0 \to R\chi_1^q \otimes_R \operatorname{Ext}_Q^1(M, N) \xrightarrow{\chi_1 \otimes \xi_1} R\chi_1^{q+1} \otimes_R \operatorname{Ext}_Q^0(M, N) \to 0.$$

We see that ${}^3\mathrm{E}^{p,\,q} = {}^\infty\mathrm{E}^{p,\,q}$ and ${}^\infty\mathrm{E}^{p,\,q} = 0$ for $(p, q) \neq (1, 2i-1), (0, 2i)$; hence

$$\operatorname{Ext}_R^n(M, N) \cong \begin{cases} \operatorname{Hom}_R(M, N) = \operatorname{Hom}_Q(M, N) & \text{for } n = 0, \\ \operatorname{Ker}\big(\xi_1\colon \operatorname{Ext}_Q^1(M, N) \to \operatorname{Hom}_Q(M, N)\big) & \text{for } n = 2i > 0, \\ \operatorname{Coker}\big(\xi_1\colon \operatorname{Ext}_Q^1(M, N) \to \operatorname{Hom}_Q(M, N)\big) & \text{for } n = 2i+1 > 0, \end{cases}$$

where the map $\chi_1\colon \operatorname{Ext}_R^n(M, N) \to \operatorname{Ext}_R^{n+2}(M, N)$ is the obvious epimorphism in degree 0 and the obvious isomorphism in positive degrees.

In view of Subsection 3.6, the preceding computation yields an effective version of Gulliksen's finiteness result, Corollary 6.2, above: if $\operatorname{Ext}_Q^p(M, N) = 0$ for $p > g + 1$ then the $R[\chi_1]$-module $\operatorname{Ext}_R^*(M, N)$ is generated in degrees $\leq g$, and χ_1 is regular on $\operatorname{Ext}_R^{>g}(M, N)$. When $\operatorname{proj\,dim}_Q M$ is finite this follows also from Subsection 2.7.

6.4.2. If $\operatorname{Ext}_Q^p(M, N) = 0$ for $p > 2$, then there is an exact sequence

$$0 \to \mathscr{H}^0\mathscr{C}^\bullet(M, N) \to \operatorname{Ext}_R^{\mathrm{even}}(M, N) \to \mathscr{H}^{-2}\mathscr{C}^\bullet(M, N)\{2\} \to 0$$

of graded $\mathscr{S}$-modules, and an isomorphism of graded $\mathscr{S}$-modules

$$\operatorname{Ext}_R^{\mathrm{odd}}(M, N) \cong \mathscr{H}^{-1}\mathscr{C}^\bullet(M, N)\{1\}.$$

When Q is local and N is its residue field k, a slightly more precise statement provided the starting point of the computation in Subsection 4.7.

7. FINITE CI-DIMENSION

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k.

Recall from [9] that a *quasi-deformation* of R (of codimension c) is a pair of local homomorphisms $R \to R' \leftarrow Q$, the first being faithfully flat and the second surjective with kernel generated by a Q-regular sequence $\boldsymbol{f}$ (of length c). An R-module M has *finite CI-dimension*, denoted $\text{CI-dim}_R M < \infty$, if the Q-module $M' = M \otimes_R R'$ has finite projective dimension for some quasi-deformation of R.

7.1

If $\mathbf{F}$ is a minimal free resolution of M over R, then $\mathbf{F} \otimes_R R'$ is one for M' over R', so $\beta_R^n(M) = \beta_{R'}^n(M')$ for all n, hence $\operatorname{cx}_R M = \operatorname{cx}_{R'} M'$. When $\operatorname{proj\,dim}_Q M'$ is finite, $\mathscr{M}' = \operatorname{Ext}^*_{R'}(M', k')$ is a finite module over $\mathscr{R}' = k'[\chi_1, \dots, \chi_c]$ by Corollary 6.2. As $\mathscr{R}'$ lives in even degrees we get a direct sum of finite graded $\mathscr{R}'$-modules

$$\operatorname{Ext}^*_{R'}(M', k') = \operatorname{Ext}^{\text{even}}_{R'}(M, k) \oplus \operatorname{Ext}^{\text{odd}}_{R'}(M', k').$$

By the Hilbert–Serre theorem, for $n \gg 0$ the sequence of even (respectively the sequence of odd) Betti numbers is given by some polynomial in n.

The following basic result (cf. [5, (3.6)] or [9, (5.10)]), provides each module of finite CI-dimension with a "best" quasi-deformation.

7.2

When $\text{CI-dim}_R M < \infty$ there exists a quasi-deformation $R \to R' \leftarrow Q$ of codimension $d = \operatorname{cx}_R M$ such that $\operatorname{proj\,dim}_Q(M \oplus_R R') < \infty$, $\operatorname{edim} Q = \operatorname{edim} R'$, and the residue field of Q is algebraically closed.

Next we give a new proof of [5, (4.1)]. As the original one, the argument presented here starts from the construction above; we believe that the route it takes from there is more enlightening (if less elementary). By convention, $0! = 1$.

7.3. THEOREM. *Let M be a finite R-module of finite CI-dimension and infinite projective dimension, and set $d = \operatorname{cx}_R M$. There exist a positive integer $\beta\text{-}\deg_R(M)$ and polynomials $p_{\text{even}}(t)$, $p_{\text{odd}}(t) \in \mathbb{Q}[t]$ of degree $\leq d-2$ such that*

$$\beta_n^R(M) = \frac{\beta\text{-}\deg_R(M)}{2^{d-1}(d-1)!}\, n^{d-1} + \begin{cases} p_{\text{even}}(n) & \text{for even } n \gg 0, \\ p_{\text{odd}}(n) & \text{for odd } n \gg 0. \end{cases}$$

We call $\beta\text{-deg}_R(M)$ the *Betti degree* of the R-module M.

7.4. EXAMPLE. In case $R = Q/(\boldsymbol{f})$ with $\operatorname{proj\,dim}_Q M = q < \infty$, let $\beta^Q(M) = \sum_{n=0}^{q} \beta_n^Q(M)$ be the *total Betti number* of M over Q. The result in [7, (3.1.3)] yields a coefficientwise inequality of formal power series

$$\sum_{n=0}^{\infty} \beta_n^R(M)\, t^n \preccurlyeq \sum_{n=0}^{q} \beta_n^Q(M)\, t^n \big/ (1-t^2)^c$$

with equality if $\boldsymbol{f} \subset \mathfrak{n}\operatorname{ann}_Q(M)$. For $2u \geq q$ the coefficient of t^{2u} on the right-hand side has the form $(b/(c-1)!)u^{c-1} + p(u)$ with $b = \sum_i \beta_{2i}^Q(M)$ and p a polynomial of degree $< c-1$. By Subsection 4.8 we have $b = \beta^Q(M)/2$; hence

$$\begin{aligned}\beta\text{-deg}_R M &= 2^{c-1}(c-1)!\lim_{u\to\infty}\frac{\beta_{2u}^R(M)}{(2u)^{c-1}} \leq \lim_{u\to\infty}\frac{\beta^Q(M)u^{c-1}+2p(u)}{2u^{c-1}}\\ &= \frac{\beta^Q(M)}{2}\end{aligned}$$

and equality holds if the regular sequence $\boldsymbol{f}$ is contained in $\mathfrak{n}\operatorname{ann}_Q(M)$.

Proof of Theorem 7.3. In view of Subsection 7.2 we may assume that $R = Q/(\boldsymbol{f})$ for a Q-regular sequence $\boldsymbol{f} = f_1, \dots, f_d$ with $d = \operatorname{cx}_R M$ and $\operatorname{proj\,dim}_Q M < \infty$. It is clear from Subsection 7.1 that d is the greater of the degrees of the polynomials q_{even} and q_{odd} that give the even and odd Betti numbers for $n \gg 0$. We have to prove that these polynomials have the same degree and equal leading coefficients.

Let $\mathscr{R} = k[\chi_1, \dots, \chi_c]$ be the ring of cohomology operators acting on $\operatorname{Ext}_R^*(M,k)$. For $r_{\text{even}} = \operatorname{rank}_{\mathscr{R}} \operatorname{Ext}_R^{\text{even}}(M,k)$ and $r_{\text{odd}} = \operatorname{rank}_{\mathscr{R}} \operatorname{Ext}_R^{\text{odd}}(M,k)$ we have

$$\begin{aligned}r_{\text{even}} &= (d-1)!\lim_{u\to\infty}\frac{\beta_{2u}^R(M)}{u^{d-1}} = 2^{d-1}(d-1)!\lim_{u\to\infty}\frac{\beta_{2u}^R(M)}{(2u)^{d-1}},\\ r_{\text{odd}} &= (d-1)!\lim_{u\to\infty}\frac{\beta_{2u+1}^R(M)}{u^{d-1}} = 2^{d-1}(d-1)!\lim_{u\to\infty}\frac{\beta_{2u+1}^R(M)}{(2u+1)^{d-1}},\end{aligned}$$

with the equalities on the left given by standard multiplicity theory. On the other hand,

$$r_{\text{even}} - r_{\text{odd}} = \operatorname{rank}_{\mathscr{R}}\Big(\bigoplus_{p\text{ even}} {}^{\infty}\mathrm{E}^{p,*}\Big) - \operatorname{rank}_{\mathscr{R}}\Big(\bigoplus_{p\text{ odd}} {}^{\infty}\mathrm{E}^{p,*}\Big) \tag{a}$$

$$= \sum_p (-1)^p \operatorname{rank}_{\mathscr{R}} {}^{\infty}\mathrm{E}^{p,*} \tag{b}$$

$$= \sum_p (-1)^p \operatorname{rank}_{\mathscr{R}} {}^1\mathrm{E}^{p,*} \tag{c}$$

$$= \sum_p (-1)^p \beta_p^Q(M) \tag{d}$$

$$= 0 \tag{e}$$

where (a) is due to the convergence of the spectral sequence of $\mathscr{R}$-modules of Theorem 6.1, (b) is due to the additivity of rank, (c) is due to the permanence of Euler characteristics, (d) is due to the complex $\mathscr{C}^\bullet(M,k)$ of Theorem 6.1, and (e) is given by Subsection 4.8. Altogether,

$$2^{d-1}(d-1)! \lim_{n\to\infty} \frac{\beta_n^R(M)}{n^{d-1}} = r_{\text{even}} = r_{\text{odd}}.$$

Thus, q_{even} and q_{odd} have equal leading term of degree d as desired. ∎

7.5. *Conjecture.* If M is a finite R-module of infinite projective dimension and finite CI-dimension with $\operatorname{cx}_R M = d$, then $\beta\text{-}\deg_R(M) \geq 2^{d-1}$, or, equivalently,

$$\lim_{n\to\infty} \frac{\beta_n^R(M)}{n^{d-1}} \geq \frac{1}{(d-1)!}.$$

7.5.1. The conjecture holds for modules of complexity ≤ 2.

Indeed, when $\operatorname{cx}_R M = 1$ the Betti numbers of M are constant for $n \gg 0$, so $\beta\text{-}\deg_R(M) = \beta_n^R(M) \geq 1$. When $\operatorname{cx}_R M = 2$ and in addition $R = Q/(f_1, f_2)$ with $\operatorname{proj\,dim}_Q M$ finite and k algebraically closed, then Subsection 4.9 yields $\lim_n \beta_n^R(M)/n = \ell/2$ and $\ell \geq 2$; Subsection 7.2 reduces the general case of complexity 2 to the special one.

7.5.2. Let M be a Q-module of finite projective dimension that is annihilated by a Q-regular sequence of length c. The squares of the elements in the given sequence form a Q-regular sequence $\boldsymbol{f}$ in $\mathfrak{n}\operatorname{ann}_R(M)$, so $\beta\text{-}\deg_R M = \beta^Q(M)/2$ by Example 7.4, and the conjecture predicts an inequality $\beta^Q(M) \geq 2^c$. The last inequality is proved in [8] when $c \leq 5$, or when M is a graded module of odd multiplicity over standard graded polynomial rings. It would follow in general from the lower bounds $\beta_n^Q(M) \geq \binom{c}{n}$ conjectured by Buchsbaum and Eisenbud [13, (1.2)].

7.5.3. If $\boldsymbol{g} = g_1, \ldots, g_c$ is a Q-regular sequence with $\boldsymbol{f} \subset \mathfrak{n}(\boldsymbol{g})$, then $M = Q/(\boldsymbol{g})$ has $\operatorname{cx}_R^M = c$ and $\beta\text{-}\deg_R M = 2^{c-1}$, so the bound in Conjecture 7.5 is sharp.

The *critical degree* $\operatorname{cr\,deg}_R M$ of a finite R-module M is defined in [9, Sect. 7] to be the infimum of those $s \in \mathbb{N}$ for which the minimal resolution $\mathbf{F}$ of M admits a chain map $\mu\colon \mathbf{F} \to \mathbf{F}$ of degree $q < 0$ with $\mu(F_{n+q}) = F_n$

for $n > s$. Clearly, $\operatorname{cr deg}_R M \le \operatorname{proj dim}_R M$ with equality if the latter is finite.

The critical degree is significant because it may be finite even when the projective dimension is not, and then it yields data on the growth of Betti numbers: when $\text{CI-dim}_R M < \infty$ and $\operatorname{depth} R - \operatorname{depth}_R M = g$ it is proved in [9, (7.3)] that

- if $\operatorname{cx}_R M \le 1$ then $\operatorname{cr deg}_R M = s \le g$ and $\beta_n^R(M) = \beta_{n+1}^R(M)$ for $n > s$,
- if $\operatorname{cx}_R M \ge 2$ then $\operatorname{cr deg}_R M = s < \infty$ and $\beta_n^R(M) < \beta_{n+1}^R(M)$ for $n > s$.

We give an *effective* bound on the critical degree for modules of finite CI-dimension and complexity 2. No bound is known in complexity ≥ 3.

7.6. THEOREM. *If M is a finite R-module with* $\text{CI-dim}_R M < \infty$ *and* $\operatorname{cx}_R M = 2$, *then* $\operatorname{cr deg}_R M \le \max\{2\beta_g^R(M) - 1, 2\beta_{g+1}^R(M)\} + g - 1$ *for* $g = \operatorname{depth} R - \operatorname{depth}_R M$.

Proof. Composition products turn $\operatorname{Ext}_R^*(k,k)$ into a graded algebra and $\operatorname{Ext}_R^*(M,k)$ into a graded left module over it. Let $\mathcal{P}$ be the k-subalgebra of $\operatorname{Ext}_R^*(k,k)$ generated by the central and primitive elements of $\operatorname{Ext}_R^2(k,k)$. By [9, (7.2)] we have to show that $\operatorname{depth}_{\mathcal{P}} \operatorname{Ext}_R^{\geqslant n}(M,k) > 0$ for $n > \max\{2\beta_g^R(M) - 2, 2\beta_{g+1}^R(M) - 1\} + g$.

Using Subsection 7.2, choose a quasi-deformation $R \to R' \leftarrow Q$ such that $R' = Q/(f_1, f_2)$ has an algebraically closed residue field k' and $M' = M \otimes_R R'$ satisfies $\operatorname{proj dim}_Q M' < \infty$. There results an equivariant isomorphism

$$\operatorname{Ext}_R^*(M,k) \otimes_k k' \cong \operatorname{Ext}_{R'}^*(M',k')$$

of graded left modules over the isomorphism of graded k'-algebras

$$\operatorname{Ext}_R^*(k,k) \otimes_k k' \cong \operatorname{Ext}_{R'}^*(k',k').$$

The last map sends $\mathcal{P} \otimes_k k'$ isomorphically onto the k'-subalgebra $\mathcal{P}'$ generated by the central and primitive elements of $\operatorname{Ext}_{R'}^2(k',k')$. As $\operatorname{Ext}_R^{\geqslant n}(M,k)$ is finite over $\mathcal{P}$ by [11, (5.3)], we have $\operatorname{depth}_{\mathcal{P}'} \operatorname{Ext}_{R'}^{\geqslant n}(M',k') = \operatorname{depth}_{\mathcal{P}} \operatorname{Ext}_R^{\geqslant n}(M,k)$, so adjusting notation we may assume that $R = Q/(f_1, f_2)$ and k is algebraically closed.

Let $\mathcal{R} = k[\chi_1, \chi_2]$ be the ring of cohomology operators acting on $\operatorname{Ext}_R^*(M,k)$. By [11, (3.3), (4.2)] there is a homomorphism of graded k-algebras $\mathcal{R} \to \mathcal{P}$ that is compatible with both actions on $\operatorname{Ext}_R^*(M,k)$. The $\mathcal{R}$-module $\operatorname{Ext}_R^{\geqslant n}(M,k)$ is finite by Corollary 6.2, so it has the same depth over $\mathcal{P}$ and over $\mathcal{R}$. The structure of the $\mathcal{R}$-module $\operatorname{Ext}_R^*(M,k)$ determined in Subsection 4.7 shows that it contains no non-zero submodule of finite length when $n > \max\{2\beta_g^R(M) - 2, 2\beta_{g+1}^R(M) - 1\} + g$. ∎

7.7

The preceding argument shows that when M is a finite R-module of finite CI-dimension the $\mathcal{P}$-module $\operatorname{Ext}_R^*(M,k)$ is generated in degree $\leq \operatorname{cr}\deg_R M + 3$. By [9, (4.9)] it is generated in the same degrees also as a module over the R-subalgebra of $\operatorname{Ext}_R^*(M,M)$ spanned by the central elements in $\operatorname{Ext}_R^2(M,M)$.

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